Université Paris-Dauphine — École doctorale de Dauphine

Solving Games and All That

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Manuscrit pour l'obtention d'un doctorat ès sciences

Spécialité informatique

8 juillet 2013

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Abstract

Efficient best-first search algorithms have been developed for deterministic two-player games with two outcomes. We present a formal framework to represent such best-first search algorithms. The framework is general enough to express popular algorithms such as Proof Number Search, Monte Carlo Tree Search, and the Product Propagation algorithm. We then show how a similar framework can be devised for two more general settings: two-player games with multiple outcomes, and the model checking problem in modal logic K. This gives rise to new Proof Number and Monte Carlo inspired search algorithms for these settings.

Similarly, the alpha-beta pruning technique is known to be very important in games with sequential actions. We propose an extension of this technique for stacked-matrix games, a generalization of zero-sum perfect information two-player games that allows simultaneous moves.

Keywords: Artificial Intelligence, Monte Carlo Tree Search, Proof Number Search, Modal Logic K, Alpha-beta Pruning

Résumé

Il existe des algorithmes en meilleur d'abord efficace pour la résolution des jeux déterministes à deux joueurs et à deux issues. Nous proposons un cadre formel pour la représentation de tels algorithms en meilleur d'abord. Le cadre est suffisamment général pour exprimer des algorithmes populaires tels Proof Number Search, Monte Carlo Tree Search, ainsi que l'algorithme Product Propagation. Nous montrons par ailleurs comment adapter ce cadre à deux situations plus générales : les jeux à deux-joueurs à plusieurs issues, et le problème de model checking en logique modale K. Cela donne lieu à de nouveaux algorithmes pour ces situations inspirées des méthodes Proof Number et Monte Carlo.

La technique de l'élagage alpha-beta est cruciale dans les jeux à actions séquentielles. Nous proposons une extension de cette technique aux *stacked-matrix games*, une généralisation des jeux à deux joueurs, à information parfaite et somme nulle qui permet des actions simultanées.

Mots clés : Intelligence Artificielle, Monte Carlo Tree Search, Proof Number Search, Logique Modale K, Élagage Alpha-beta

Contents

Acknowledgments

My doctoral studies have come to a happy conclusion. As I contemplate the journey, I see that a score of people have had a distinct positive influence on my last three years. I cannot list the name of every single person that I am grateful to, but I will try to exhibit a representative sample.

Tristan Cazenave, you are passionate about games and artificial intelligence, hard-working, and yet you manage to lead a balanced life. More than a research advisor, you have been a role model. The more I observe you, the more I know I want to be an academic.

Stefan Edelkamp and Olivier Teytaud, you immediately agreed to review my thesis despite your busy schedule and you provided valuable feedback. Discussing with you has always been enlightening and I look forward to starting working with you. Andreas Herzig and Martin Müller, you accepted to serve on my thesis committee. I am also thankful for the short and long research visits that I made in your respective groups.

Mohamed-Ali Aloulou, Cristina Bazgan, Denis Bouyssou, Virginie Gabrel, Vangelis Paschos, and Alexis Tsoukiàs, be it for books, for conferences, research visits, or even to go spend a year in another university on another continent, I was always generously supported.

Caroline Farge, Valérie Lamauve, Mireille Le Barbier, Christine Vermont, and the other staff from University Paris-Dauphine, everyday, you build bridges between the bureaucracy and the absent-minded researchers and students so that the administrative labyrinth becomes less of a hassle. In particular Katerina Kinta, Nathalie Paul de la Neuville, and Olivier Rouyer, you always spent the time needed to solve my daily riddles, even when the situation was complicated

or didn't make sense.

Jérôme Lang, you introduced me to another research community and acted as a mentor giving me advice and answering my many questions about the research process. Hans van Ditmarsch, from the day we met, you treated me like a colleague rather than a student, it surely helped me gain confidence. Flavien Balbo, Édouard Bonnet, Denis Cornaz, and Suzanne Pinson, you trusted me and teaching for/with you was a real pleasure.

Michael Bowling, Michael Buro, Ryan Hayward, Mike Johanson, Martin Müller, Rick Valenzano and the members of the Hex, Heuristic Search, Monte Carlo, Poker, and Skat Meeting Groups, with you I benefited from an unlimited supply of ideas and insights.

My coauthors, Chris Archibald, Édouard Bonnet, Michael Buro, Cédric Buron, Tristan Cazenave, Dave Churchill, Hans van Ditmarsch, Edith Elkind, Hilmar Finnsson, Florian Jamain, Nicolas Jouandeau, Marc Lanctot, Jérôme Lang, Jean Méhat, Michael Schofield, Joel Veness, and Mark Winands, as well as other people I have collaborated with, for your creative input.

Nathanaël Barrot, Amel Benhamiche, Morgan Chopin, Miguel Couceiro, Tom Denat, Eunjung Kim, Renaud Lacour, Dalal Madakat, Mohamed Amine Mouhoub, Nicolas Paget, Lydia Tlilane, and the other CS faculty, PhD students, interns, and alumni in Dauphine, you all contributed to a friendly and welcoming atmosphere in our workplace. Émeric Tourniaire, you recognize that form matters and you were always ready to help improve my presentation and typography skills. Raouia Taktak, somehow it always was comforting to meet you in the hallway in the middle of the night as we both tried to finish writing our respective dissertations.

Anthony Bosse and Camille Crépon, you were there in the bad times and you were there in the good times, I am lucky to count you among my friends. Vincent Nimal, you always were in an another country but I knew I could reach you and talk to you any time I wanted. Sarah Juma, with your patience and understanding, I have learned more than any higher education can provide.

Marc Bellemare, thank you for the welcome you gave me in Edmonton when I arrived in your country and in Québec when I was about to leave it, you've been so helpful all the way long. James, Mohsen, and Nikos, living, drinking, cooking, and watching movies with you guys made Canada feel home despite the cold winter.

Felicity Allen, Marc Bellemare, Édouard Bonnet, Dave Churchill, Tim Furtak, Richard Gibson, Florian Jamain, Marc Lanctot, Arpad Rimmel, Fabien Teytaud, Joel Veness, you were always ready to play, be it abstract games, board games, card games, or video games. Michael Buro, Rob Holte, and Jonathan Schaeffer, the GAMES parties you organized are among my favorite memories of Edmonton.

Bonita Akai, Eric Smith, and the members of the University of Alberta Improv Group, it was a lot of fun to spend time with you and you definitely contributed to the balance of my Canadian life. Pierre Puy, Richard Soudée, and the member of the Association de Théâtre à Dauphine, I enjoyed spending those Friday evenings and nights with you.

Pilar Billiet and Maté Rabinovski, whenever I needed to escape my Parisian routine, you offered me a quiet, cultural, and gastronomical break. Thérèse Baskoff, Hélène and Matthieu Brochard, Danièle Geesen, and Monique Nybelen, without you, the final day of my PhD would have been much less enjoyable. Thank you for your help.

My grand-parents, Grand-Mère, for your jokes, always unexpected and creative, and Grand-Père, for your experienced input and sharp guidance, always relevant. Mum and Dad, you always cared to make your work and projects accessible to other people. Thank you for showing me that there is much fun to be found at work. Sonya, my sister, with such a positive bias in your eyes, I can only feel better when you talk to me.

Finally, I would also like to express a thought for Hamid, Nabila, Pierrot, Thierry, and the rest of my family whom I haven't seen much in the last few years, for Jonathan Protzenko, Sébastien Tavenas, Jorick Vanbeselaere, Sébastien Wémama, and my other friends, for those too few but cheering and joyful times.

1 Introduction

1.1 Motivation

The term *multi-agent system* has been used in many different situations and it does not correspond to a single unified formalism. Indeed, formal concepts such as extensive-form games, multi-agent environments, or Kripke structures can all be thought of as referring to some kind of multi-agent system.

A large fraction of the multi-agent systems lend themselves to a multi-stage interpretation. This multi-stage interpretation is non only relevant in domains where agents perform actions sequentially, but also, say, in epistemic logics where agents can have higher order knowledge/beliefs or perform introspection. The underlying structure of these multi-stage problems is that of a graph where the vertex correspond to states of the world and the edges correspond to the actions the agents can take or to the applications of modal operators in epistemic logic.

Properties of the system can thus be reduced to properties of the underlying graph. The algorithmic stance adopted in this thesis consists of expressing concrete heuristics or algorithms that allow to understand a multi-agent system through the exploration of the corresponding graph. In most non-trivial such multi-agent systems, the underlying graph is too large to be kept in memory and explored fully. This consideration gives rise to the *Search* paradigm. In *Search*, the graph is represented implicitly, typically with some starting vertex and a successor function that returns edges or vertices adjacent to its argument.

Opposite to this high-level description of search problems in general, we have a variety of concrete applications, research communities, and, accordingly,

typical assumptions on the graphs of interest. As a result, many classical search algorithms are developed with these assumptions in mind and seem to be tailored to a specific class of multi-agent systems. The conducting line of our work is to study whether and how such algorithms can be generalized and some assumptions lifted so as to encompass a larger class of multi-agent systems.

The research presented in this thesis has two main starting points, the alpha-pruning technique for the depth-first search algorithm known as *minimax* on the one hand, and the Monte Carlo Tree Search (MCTS) and Proof Number Search (PNS) algorithms on the other hand.

The minimax algorithm which is a generalization of *depth-first search* to sequential two-player zero-sum games can be significantly improved by the *alpha-beta pruning* technique. Alpha-beta pruning avoids searching subtrees which are provably not needed to solve the problem at hand. Two important facts contribute to the popularity of alpha-beta pruning in game search. It is a safe pruning technique in that the result returned by the depth-first search is not affected when pruning is enabled. Discarding subtrees according to the alpha-beta pruning criterion can lead to considerable savings in terms of running time. Indeed, Knuth and Moore have shown that if a uniform tree of size n was explored by the minimax algorithm, alpha-beta pruning might only necessitate the exploration of a subtree of size $\sqrt{n}$.

Alpha-beta pruning contributed to the creating of very strong artificial players in numerous games from CHESS to OTHELLO. However, the original algorithm for alpha-beta pruning only applied to deterministic sequential zero-sum two-player games of perfect information (called *multi-outcome games* in this thesis, see Chapter 3). This is quite a strong restriction indeed and there have been many attempts at broadening the class of multi-agent systems that can benefit from alpha-beta-like safe pruning. Ballard and Hauk et al. have shown how to relax the deterministic assumption so that safe pruning could be applied to stochastic sequential zero-sum two-player games of perfect information [13, 61]. Sturtevant has then shown how the two-player and the zero-sum assumptions could be alleviated [147, 148]. In Chapter 5, we lift the sequentiality assumption and show how safe alpha-beta-style pruning can be performed in zero-sum two-player games with simultaneous moves. Thus, two tasks remain to be completed before safe alpha-beta pruning can be applied to a truly general class of multi-agent system. Creating a unified formalism that

would allow combining the aforementioned techniques and providing pruning criteria for imperfect information games in extensive-form.

The PNS and MCTS algorithms were first suggested as ways to respectively solve and play deterministic sequential two-player Win/Loss games of perfect information (called *two-outcome games* in this thesis, see Chapter 2). Both algorithms proved very successful at their original tasks. Variants of PNS [74] were essential to solve a number of games, among which CHECKERS [136], FANORONA [131], as well as medium sizes of HEX [8]. On the other hand, the invention of the Upper Confidence bound for Trees (UCT) [76] and MCTS [40] algorithms paved the way for the *Monte Carlo revolution* that improved considerably the computer playing level in a number of games, including GO [85], HEX [7], and General Game Playing (GGP) [47] (see the recent survey by Browne et al. for an overview [20]).

Besides actual games, these algorithms have been used in other settings that can be represented under a similar formalism, notably chemical synthesis [64] and energy management problems [39].

In their simplest form, the PNS and MCTS algorithms maintain a partial game tree in memory and they share another important feature. They can be both expressed as the iteration of the following four-step process: descend the tree until a leaf is reached, expand the leaf, collect some information on the new generated leaves, backpropagate this information in the tree up to the root.

This leads us to define a Best First Search (BFS) framework consisting exactly of these four steps and parameterized by an *information scheme*. The information scheme determines the precise way the tree is to be traverse, the kind of information collected at leaves and how information is backpropagated. The BFS framework is first defined in Chapter 2 for two-outcome games and then extended to multi-outcome games and to Multi-agent Modal Logic K (MMLK) model checking.

1.2 Organization and Contributions

The common formalism used throughout this thesis is the *transition system* (see Definition 1 in Section 1.4). Transition systems have been used in a variety of domains, and particularly in verification and model checking [11]. In this thesis, we shall focus on a few selected classes of multi-agent systems for which

we will present and develop appropriate solving techniques. Each chapter of this thesis is organized around a specific class and we will see how they can all be viewed as particular transition systems where a few additional assumptions hold.

Chapter 2 Two-player two-outcome games

Chapter 3 Two-player multi-outcome games

Chapter 4 Models of Multi-agent Modal Logic K

Chapter 5 Stacked-matrix games

More precisely, the contributions presented in this thesis include

- – A formal BFS framework for two-outcome games based on the new concept of information scheme;
  - – information schemes generating the PNS, MCTS Solver, and Product Propagation (PP) algorithms;
  - – an experimental investigation of PP demonstrating that PP can sometimes perform significantly better than the better known algorithms PNS and MCTS;
- – an extension of the BFS framework to multi-outcome games through the new concept of multi-outcome information scheme;
  - – an information scheme defining the Score Bounded Monte Carlo Tree Search (SBMCTS) algorithm, a generalization of MCTS Solver;
  - – a principled approach to transforming a two-outcome information scheme into a multi-outcome information scheme;
  - – the application of this approach to develop Multiple-Outcome Proof Number Search (MOPNS), a generalization of PNS to multi-outcome games and an experimental study of MOPNS;
- – an adaptation of the proposed BFS framework to the model checking problem in MMLK, yielding several new model checking algorithms for MMLK;

  - Minimal Proof Search (MPS), an optimal algorithm to find (dis)proofs of minimal size for the model checking problem in MMLK.
  - a formal definition of many solution concepts popular in sequential games via MMLK formula classes, including *ladders* in two-player games, and *paranoid wins* in multi-player games;
  - the use of MMLK reasoning to prove formal properties about these solution concepts and to provide a classification of number of algorithms for sequential games;

- – a generalization of Alpha-Beta pruning in games with simultaneous moves, Simultaneous Move Alpha-Beta (SMAB);
  - an efficient heuristic algorithm for games with simultaneous moves under tight time constraints in the domain of Real-Time Strategy (RTS) games, Alpha-Beta (Considering Durations) (ABCD);
  - an experimental investigation of these new algorithms.

1.3 Contributions not detailed in this thesis

1.3.1 Endgames and retrograde analysis

The algorithms presented in this thesis are based on forward search. Given an initial state s_0, they try to compute some property of s_0, typically its game theoretic value, by examining states that can be reached from s_0.

It is sometimes possible to statically, i.e., without search, compute the game theoretic value of a game position even though it might no be a final position. We developed a domain specific technique for the game of BREAKTHROUGH called *race patterns* that allows to compute the winner of positions that might need a dozen additional moves before the winner can reach a final state [126]. We also proposed a parallelization of the PN^2 algorithm on a distributed system in a fashion reminiscent of Job-Level Proof Number Search [168]. An implementation of race patterns and the parallelization of PN^2 on a 64-client system allowed us to solve BREAKTHROUGH positions up to size 6×5 while the largest position solved before was 5×5.

An interesting characteristic of a number of domains that we try to solve is that they are *convergent*, that is, there are few states in the endgames compared

to the middle game. For example, CHESS is convergent as the number of possible states shrinks as the number of pieces on the board decreases. It is possible to take advantage of this characteristic by building endgame databases that store the game theoretic value of endgame positions that have been precomputed. In CHESS, endgame databases, or rather one particularly efficient encoding called *Nalimov tables*, are now pervasive and used by every competitive CHESS playing engine [152, 104]. Endgame databases have been crucial to solving other games such as CHECKERS [135], AWARI [116], and FANORONA [131].

An endgame database does actually not need to contain all endgame positions but only a representative position for every symmetry equivalence class. Geometrical symmetry is the most common type of symmetry and it typically involves flipping or rotating the game board [42]. Another kind of symmetry occur in trick-taking card games, where different cards can take corresponding roles. We call this it *material symmetry* and we show argue that it occurs in a variety of games besides trick-taking card games.

We argue that material symmetry can often be detected via the graph representing the possible interaction of the different game elements (the material) [128]. Indeed, we show in three different games, SKAT, DOMINOES, and CHINESE DARK CHESS that material symmetry reduces to the subgraph isomorphism problem in the corresponding interaction graph. Our method yields a principled and relatively domain-agnostic approach to detecting material symmetry that can leverage graph theory research [154]. While creating a domain-specific algorithm for detecting material symmetry in SKAT and DOMINOES is not hard, interactions between pieces in CHINESE DARK CHESS are quite intricate and earlier work on CHINESE DARK CHESS discarded any material symmetry. On the other hand, the interaction graph follows directly from the rules of the game and we show that material symmetry can lead to equivalent databases that are an order of magnitude smaller.

> [126] Abdallah Saffidine, Nicolas Jouandeau, and Tristan Cazenave. Solving Breakthough with race patterns and Job-Level Proof Number Search. In H. van den Herik and Aske Plaat, editors, *Advances in Computer Games*, volume 7168 of *Lecture Notes in Computer Science*, pages 196–207. Springer-Verlag, Berlin / Heidelberg, November 2011. ISBN 978-3-642-31865-8. doi: 10.1007/978-3-642-31866-5_17

[128] Abdallah Saffidine, Nicolas Jouandeau, Cédric Buron, and Tristan Cazenave. Material symmetry to partition endgame tables. In *8th International Conference on Computers and Games (CG)*. Yokohama, Japan, August 2013

1.3.2 Monte Carlo Methods

Monte Carlo methods are being more and more used for game tree search. Besides the Score Bounded Monte Carlo Tree Search algorithm that we detail in Chapter 3, we have investigated two aspects of these Monte Carlo methods. In a first line of work, we focused on the MCTS algorithm and studied how transpositions could be taken into account [125]. After showing a few theoretical shortcomings of some naive approaches to handling transpositions, we proposed a parameterized model to use transposition information. The parameter space of our model is general enough to represent the naive approach used in most implementations of the MCTS algorithm, the alternative algorithms proposed by Childs et al. [27], as well a whole range of new settings. In an extensive experimental study ranging over a dozen domains we show that it is consistently possible to improve upon the standard way of dealing with transposition. That is, we show that the parameter setting simulating the standard approaches almost always perform significantly worse than the optimal parameter setting.

In a second line of work, we propose a new Monte Carlo algorithm for stochastic two-player games with a high branching factor at chance nodes [83]. The algorithms we propose are quite similar to EXPECTIMAX and its pruning variants STAR1 and STAR2 [61]. The only difference is that instead of looping over all possible moves at a chance nodes, we sample a bounded subset of moves. This allows searching faster or much deeper trees at the cost of some inaccuracy in the computed value. We show that the computed value is accurate with a high probability that does not depend on the true branching factor at chance nodes. This results constitute a generalization of *sparse sampling* from Markov Decision Processes to stochastic adversarial games [72]. It can also be related to the double progressive widening idea [38]. We conduct an experimental study on four games and show that the new approach consistently outperforms their non-sampling counterparts.

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Confidence bound for rooted Directed acyclic graphs. *Knowledge-Based Systems*, 34:26–33, December 2011. doi: 10.1016/j.knosys.2011.11.014

[83] Marc Lanctot, Abdallah Saffidine, Joel Veness, Chris Archibald, and Mark Winands. Monte carlo *-minimax search. In *23rd International Joint Conference on Artificial Intelligence (IJCAI)*, Beijing, China, August 2013. AAAI Press

1.3.3 Analysis of the Game Description Language

The formalism use throughout this thesis is based on transition systems. These transition systems notably include a state space and a transition relation. However, in practice the state space is implicit and uses a domain specific *state representation*. In that case, the transition relation is given by domain specific *game rules*.

The most straightforward approach to running concrete algorithms on a domain, is to implement the mechanics of the domain directly in some programming language and to provide procedure to manipulate states in a specified interface. The algorithms to be tested are implemented in the same programming language and can be adapted to use the specified interface.

One downside to this approach is that describing game rules in the same procedural language as the algorithms might be tedious for some games. Even more so, this approach makes automatic comparison between competing algorithms implemented by different people rather difficult. Indeed, when we compare two implementations of two competing algorithms based on two different implementation of the domain, determining whether a speed-up is due to an improvement on the algorithm side or is due to a domain-specific trick is usually tricky, particularly when the implementations are not publicly available.

An alternative approach is to develop a standard modeling language to represent domains and then have interfaces from the language of the domains to the programming language of the algorithms. We can then measure the merits of various algorithms on the very same domain without fear that domain specific improvements might creep in some implementations only.

This idea was successfully brought into effect in multiple research communities. For instance, the PROMELA language was designed to represent distributed system and makes it possible to implement model checking or verification

algorithms in a domain agnostic way [67, 68]. In planning, the Planning Domain Description Language (PDDL) was developed to be used as part of the international planning competitions [48, 65].

In the games community, the Game Description Language (GDL) was introduced to model a large variety of multi-agent transition systems [93]. GDL was used as a domain language in the yearly GGP competition and hundreds of games have been specified in this language. Interfacing domain written in GDL with a game playing engine is traditionally based on a Prolog interpreter such as YAP [37], and Prolog bindings in the programming language of the playing engine.

A few other methods have since been suggested to deal with GDL and provide the needed interface. For instance, under some assumptions, it is possible to ground the game rules and use an Answer-Set Programming solver [151, 101] to determine legal transitions or even solve some single-agent instances. We have proposed an compiling approach to GDL based on forward chaining [121]. The compilation strategy is based on successive program transformations that have proved successful in other domains (notably the Glasgow Haskell Compiler [96] and the CompCert C compiler [87]). The forward chaining approach we use is adapted from the Datalog interpretation scheme advocated by Liu and Stoller [89], but we outlined a few optimizations specific to GDL.

Most compilers and interpreters for GDL actually only support a subset of the language. This is not a shortcoming unbeknownst to the authors of the said systems but rather a design choice. These implementations indeed impose restrictions on GDL to allow for further optimizations at the cost of not handling a small subset of the games appearing in practice. A popular such restriction is to prevent nested function constants in terms, or at least to have bounded nesting depth which is the case for the vast majority of GGP games used in international competitions. We motivate formally this design choice by showing that the full Game Description Language is Turing complete [120]. As a consequence many properties of GDL rules are undecidable. Bounding the nesting depth (as well as other typical restrictions) make these properties decidable.

More recently, we have improved the forward chaining compilation of GDL in a new implementation that introduces an additional set of lower-level optimizations, leading to very efficient generated code [140].

[121] Abdallah Saffidine and Tristan Cazenave. A forward chaining

based game description language compiler. In *IJCAI Workshop on General Intelligence in Game-Playing Agents (GIGA)*, pages 69–75, Barcelona, Spain, July 2011

[120] Abdallah Saffidine. The Game Description Language is Turing-complete. *IEEE Transactions on Computational Intelligence and AI in Games*, 2013. submitted

[140] Michael Schofield and Abdallah Saffidine. High speed forward chaining for general game playing. In *IJCAI Workshop on General Intelligence in Game-Playing Agents (GIGA)*, Beijing, China, August 2013. submitted

1.3.4 Complexity of Solving Games

Multiple approaches for solving games are presented in this thesis, but all of them rely on an explicit exploration of at least a fraction of the state space. Since the state space can be implicitly represented, e.g., when the game is specified in the GDL (see Section 1.3.3), the state space is usually exponentially bigger than the domain specific representation of a state.

As a result, the algorithms we advocate typically are exponential in the size of the input. Since they can in principle solve games of any size, they are particularly adapted to games that are computationally complex, as polynomial algorithms for such games are unlikely.

Determining the computational complexity of generalized version of games is a popular research topic [63]. The complexity class of the most famous games such as CHECKERS, CHESS, GO was established shortly after the very definition of the corresponding class [50, 49, 115]. Since then, other interesting games have been classified, including OTHELLO [71] and AMAZONS [53]. Reisch proved the PSPACE completeness of the most famous connection game, HEX, in the early 80s [113]. We have since then proved that HAVANNAH and TWIXT, two other notable connection games, were PSPACE-complete [15].

Trick-taking card games encompass classics such CONTRACT BRIDGE, SKAT, HEARTS, SPADES, TAROT, and WHIST as well as hundreds of more exotic variants.[1]

[1] A detailed description of these games and many other can be found on.`http://www.pagat.com/class/trick.html`.

A significant body of Artificial Intelligence (AI) research has studied trick-taking card games [22, 57, 51, 80, 94] and Perfect Information Monte Carlo (PIMC) sampling is now used as a base component of virtually every state-of-the-art trick-taking game engine [88, 57, 150, 91]. Given that most techniques based on PIMC sampling rely on solving perfect information instance of such trick-taking games, establishing complexity of the perfect information variant of these games is a pressing issue.

Despite their huge popularity in the general population as well as among researchers, BRIDGE and other trick-taking card games remained for a long time virtually unaffected by the stream of complexity results on games. In his thesis, Hearn proposed the following explanation to the standing lack of hardness result for such games [62, p122].

> There is no natural geometric structure to exploit in BRIDGE as there is in a typical board game.

In a recent paper [16], we propose a general model for perfect information trick-taking card games and prove that solving an instance is PSPACE-complete. The model can be restricted along many dimensions, including the number of suits, the number of players, and the number of cards per suit. This allows to define fragments of the class of trick-taking card games and it makes it possible to study where the hardness comes from. In particular, tractability results by Wästlund fall within the framework [163, 164]. We also show that bounding the number of players or bounding the number of cards per suit is not sufficient to avoid PSPACE-hardness. The results of the paper can be summed up in the complexity landscape of Figure 1.1.

> [15] Édouard Bonnet, Florian Jamain, and Abdallah Saffidine. Havannah and Twixt are PSPACE-complete. In *8th International Conference on Computers and Games (CG)*. Yokohama, Japan, August 2013
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> [16] Édouard Bonnet, Florian Jamain, and Abdallah Saffidine. On the complexity of trick-taking card games. In *23rd International Joint Conference on Artificial Intelligence (IJCAI)*, Beijing, China, August 2013. AAAI Press

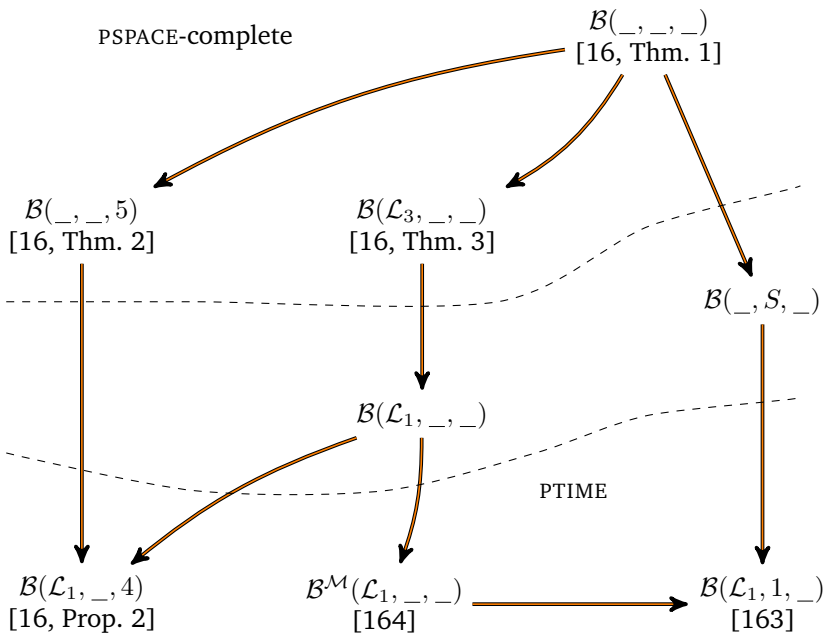


Figure 1.1: Summary of the hardness and tractability results known for the fragments of the class of trick-taking card games $\mathcal{B}(\mathcal{L}, s, l)$. An underscore _ means that the parameter is not constrained. In the first parameter, $\mathcal{L}_i$ indicates that there are $2i$ players partitioned in two team of equal size. The second parameter is the number of suits s, and the third parameter is the maximum number of cards per suit. Finally, $\mathcal{B}^{\mathcal{M}}(,,)$ indicates a symmetry restriction on the distribution of suits among players.

1.3.5 Computational Social Choice

If studying algorithms that compute Nash equilibria and other solution concepts in specific situations constitute one end of the multi-agent/algorithmic game theory spectrum, then computational social choice can be seen as the other end of the spectrum. In computational social choice, one is indeed interested in solution concepts in their generality. Typical computational social choice investigations include the following questions.

- What properties does a particular solution concept have?
- Is there a solution concept satisfying a given list of axiom?
- Is the computation of a given property in a given class of multi-agent systems tractable?
- Can we define a class of multi-agent systems that could approximate a given *real-life* interaction among agents?
- If so, what new solution concepts are relevant in the proposed class and how do they relate to existing solution concepts in previously defined classes?

A subfield of computational social choice of special interest to us is that of elections. Elections occur in multiple real-life situations and insight from social choice theory can be fruitfully applied to settings as varied as political elections, deciding which movie a group of friend should watch, or even selecting a subset of submissions to be presented at a conference. Another setting closer to the main topic of this thesis can also benefit from social choice insights: ensemble based decision making [112] has recently been successfully applied to the games of SHOGI [107] and GO [95] via a majority voting system.

Given a set of candidates and a set of voters, a preference profile is a mapping from each voter to a linear order on the candidates. A *voting rule* maps a preference profile to an elected candidate. It is also possible to define voting rules that map preference profiles to sets of elected candidates. Social choice theorist study abstract properties of voting rules to understand which rule is more appropriate to which situation. We refer to Moulin's book for a detailed treatment of the field [99].

An very important solution concept in single winner elections is that of a *Condorcet winner*. A Condorcet winner is a candidate that is preferred by majority of voters to every other candidates in one-to-one elections. A Condorcet winner does not always exists for a given preference profile, but when one does, it is reasonable to expect that it should be elected. We proposed a generalization of the Condorcet winner principle to multiple winner elections [43]. We say that a set of candidates is a *Condorcet winning set* if no other candidate is preferred to all candidates in the set by a majority of voter. Just as Condorcet winners, Condorcet winning sets satisfy a number of desirable social choice properties. Just as Condorcet winners, Condorcet winning sets of size 2 are not guaranteed to exist and we ask whether for any size k, there exists a profile P_k such that P_k does not admit any Condorcet winning set of size k.

Another line of work that we have started exploring deals with voters' knowledge of the profile [160]. The fact that voters may or may not know each other's linear order on the candidate has multiple consequences, for instance on the possibilities of manipulation. We propose a model based on epistemic logic that accounts for uncertainty that voters may have about the profile. This model makes it possible to model higher-order knowledge, e.g., we can model that a voter v_1 does not know the preference of another voter v_3, but that v_1 knows that yet another voter v_2 knows v_3's linear order.

[43] Edith Elkind, Jérôme Lang, and Abdallah Saffidine. Choosing collectively optimal sets of alternatives based on the Condorcet criterion. In Toby Walsh, editor, *22nd International Joint Conference on Artificial Intelligence (IJCAI)*, pages 186–191, Barcelona, Spain, July 2011. AAAI Press. ISBN 978-1-57735-516-8

[160] Hans van Ditmarsch, Jérôme Lang, and Abdallah Saffidine. Strategic voting and the logic of knowledge. In Burkhard C. Schipper, editor, *14th conference on Theoretical Aspects of Rationality and Knowledge (TARK)*, pages 196–205, Chennai, India, January 2013. ISBN 978-0-615-74716-3

1.4 Basic Notions and Notations

We now introduce a few definitions and notations that we will use throughout the thesis.

Definition 1. A *transition system* T is a tuple $\langle S, R, \rightarrow, L, \lambda \rangle$ such that

- S is a set of *states*;
- R is a set of *transition labels*;
- $\rightarrow \subseteq S \times R \times S$ is a transition relation;
- L is a set of *state labels*;
- $\lambda : S \rightarrow 2^L$ is a labeling function. This function associate a set of labels to each state.

For two states $s, s' \in S$ and a transition label $a \in R$, we write $s \xrightarrow{a} s'$ instead of $(s, a, s') \in \rightarrow$. If s is a bound state variable, we indulge in writing $\exists s \xrightarrow{a} s'$ instead of $\exists s' \in S, s \xrightarrow{a} s'$. Similarly, if s' is bound, the same notation $\exists s \xrightarrow{a} s'$ means $\exists s \in S, s \xrightarrow{a} s'$. In the same way, we allow the shortcut $\forall s \xrightarrow{a} s'$.

We recall that *multisets* are a generalization of sets where elements are allowed to appear multiple times. If A is a set, then 2^A denotes the *power set* of A, that is, the set of all sets made with elements taken from A. Let $\mathbb{N}^A$ denote the set of multisets of A, that is, the set of all multisets made with elements taken from A. We denote the *carrier* of a multiset M by M^*, that is M^* is the set of all elements appearing in M.

We recall that a *total preorder* on a set is a total, reflexive, and transitive binary relation. Let A be a set and $\preccurlyeq$ a total preorder on A. $\preccurlyeq$ is total so every pair of elements are in relation: $\forall a, b \in A$ we have $a \preccurlyeq b$ or $b \preccurlyeq a$. $\preccurlyeq$ is reflexive so every element is in relation with itself: $\forall a \in A$ we have $a \preccurlyeq a$. $\preccurlyeq$ is transitive so $\forall a, b, c \in A$ we have $a \preccurlyeq b$ and $b \preccurlyeq c$ implies $a \preccurlyeq c$.

Basically, a total preorder can be seen as a total order relation where distinct elements can be on the "same level". It is possible to have $a \neq b$, $a \preccurlyeq b$, and $b \preccurlyeq a$ at the same time.

We extend the notation to allow comparing sets. If $\preccurlyeq$ is a total preorder on A and A_1 and A_2 are two subsets of A, we write $A_1 \preccurlyeq A_2$ when $\forall a_1 \in A_1, \forall a_2 \in A_2, a_1 \preccurlyeq a_2$.

We also extend the notation to have a strict preorder: $a \prec b$ if and only if $a \preccurlyeq b$ and $b \not\preccurlyeq a$. Finally, we extend the strict notation to allow comparing sets, we write $A_1 \prec A_2$ when $\forall a_1 \in A_1, \forall a_2 \in A_2, a_1 \prec a_2$.

2 Two-Outcome Games

We define a formal model of deterministic two-player perfect information two-outcome games. We develop a generic best-first-search framework for such two-outcome games and prove several properties of this class of best-first-search algorithms. The properties that we obtain include correctness, progression, and completeness in finite acyclic games. We show that multiple standard algorithms fall within the framework, including PNS*,* MCTS*, and* PP*.*

The Chapter includes results from the following paper.

[124] Abdallah Saffidine and Tristan Cazenave. Developments on product propagation. In 8th International Conference on Computers and Games (CG)*. Yokohama, Japan, August 2013*

Contents

2.1 Game Model

We base the definition of two-outcome games on that of transition system (see Definition 1). Transition labels are interpreted as *agents* or *players*.

Definition 2. A *two-outcome game* is a transition system $\langle S, R, \rightarrow, L, \lambda \rangle$ where the following restriction holds.

- There are two distinguished agents *Max* $\in R$ and *Min* $\in R$
- State turns are exclusive: $\neg\exists s_1, s_2, s_3 \in S, s_1 \xrightarrow{Max} s_2 \wedge s_1 \xrightarrow{Min} s_3$.
- There is a distinguished label: *Win* $\in L$;

We define the *max states* A and the *min states* B as the sets of states that allow respectively *Max* and *Min* transitions: $A = \{s \in S, \exists s' \in S, s \xrightarrow{Max} s'\}$ and $B = \{s \in S, \exists s' \in S, s \xrightarrow{Min} s'\}$.

We say that a state is *final* if it allows no transition for *Max* nor *Min*. We denote the set of final states by F. $F = S \setminus (A \cup B)$. States that are not final are called *internal*. For two states $s_1, s_2 \in S$, we say that s_2 is a *successor* of s_1 if it can be reached by a *Max* or a *Min* transition. Formally, we write $s_1 \rightarrow s_2$ when $s_1 \xrightarrow{Max} s_2 \vee s_1 \xrightarrow{Min} s_2$.

Remark 1. *From the turn exclusivity assumption, we derive that A, B, and F constitute a partition of S.*

We say that a state is *won*, if it is final and it is labelled as a *Win*: $s \in F \wedge$ *Win* $\in \lambda(s)$. We say that a state is *lost* if it is final and it is not won.

Note that we have not mentionned any other agent beside *Max* and *Min* nor any state label besides *Win*. Other agents and other state labels will have no influence in this Chapter and we will assume without loss of generality that $R = \{Max, Min\}$ and $L = \{Win\}$.

The game graph is a Direct Acyclic Graph (DAG) if there are no sequences $s_0 \to s_1 \to \ldots \to s_k \to s_0$. When the game graph is a finite DAG, we can define the height h of a state to be the maximal distance from that state to a final state. If $s \in F$ then $h(s) = 0$ and if $s \in A \cup B$ then $h(s) = 1 + \max_{s \to s'} h(s')$.

Definition 3. A *weak* Max-*solution* to a two-outcome game is a subset of states $\Sigma \subseteq S$ such that

$$\text{If } s \in F \text{ then } s \in \Sigma \Rightarrow \textit{Win} \in \lambda(s) \tag{2.1}$$

$$\text{If } s \in A \text{ then } s \in \Sigma \Rightarrow \exists s \xrightarrow{\textit{Max}} s', s' \in \Sigma \tag{2.2}$$

$$\text{If } s \in B \text{ then } s \in \Sigma \Rightarrow \forall s \xrightarrow{\textit{Min}} s', s' \in \Sigma \tag{2.3}$$

Conversely, a *weak* Min-*solution* is a subset of states $\Sigma \subseteq S$ such that

$$\text{If } s \in F \text{ then } s \in \Sigma \Rightarrow \textit{Win} \notin \lambda(s) \tag{2.4}$$

$$\text{If } s \in A \text{ then } s \in \Sigma \Rightarrow \forall s \xrightarrow{\textit{Max}} s', s' \in \Sigma \tag{2.5}$$

$$\text{If } s \in B \text{ then } s \in \Sigma \Rightarrow \exists s \xrightarrow{\textit{Min}} s', s' \in \Sigma \tag{2.6}$$

Remark 2. *Weak* Max-*solutions on the one hand, and weak* Min-*solutions on the other hand are closed under union but are not closed under intersection.*

The class of systems that we focus on in this Chapter and in Chapter 3 are called *zero-sum* games. It means that the goals of the two players are literally opposed. A possible understanding of the zero-sum concept in the proposed formalism for two-outcome games is that each state is either part of some weak *Max*-solution, or part of some weak *Min*-solution, but not both.

Proposition 1. *Let Σ be a weak* Max-*solution and Σ' be a weak* Min-*solution. If the game graph is a finite DAG, then these solutions do not intersect: $\Sigma \cap \Sigma' = \emptyset$.*

Proof. Since the game graph is a finite DAG, the height of states is well defined. We prove the proposition by induction on the height of states.

Base case: if a state s has height $h(s) = 0$, then it is a final state. If it is part of the *Max*-solution, $s \in \Sigma$, then we know it has label *Win*, $\textit{Win} \in \lambda(s)$ and it cannot be in the weak *Min*-solution.

Induction case: assume there is no state of height less or equal to n in $\Sigma \cap \Sigma'$ and obtain there no state of height $n + 1$ in $\Sigma \cap \Sigma'$. Let us take $s \in \Sigma$ such

that $h(s) = n + 1$ and prove that $s \notin \Sigma'$. If $s \in A$ then by definition of a weak *Max*-solution s has a successor $c \in \Sigma$. Since all successors of s have height less or equal to n, we know that $h(c) \leq n$. From the induction hypothesis, we obtain that c is not in Σ'. Hence, s cannot be in Σ' either as it would require all successors and c in particular to be in Σ'. □

Proposition 2. *Let s be a state, if the game graph is a finite DAG, then s belongs to a weak solution.*

Proof. Since the game graph is a finite DAG, the height of states is well defined. We prove the proposition by induction on the height of states.

Base case: if a state s has height $h(s) = 0$, then it is a final state. If it has label *Win*, then we known s is part of a *Max*-solution, for instance $\Sigma = \{s\}$. Otherwise, it is part of a *Min*-solution, for instance $\Sigma = \{s\}$.

Induction case: assume all states of height less or equal to n are part of a weak solution and obtain that any state of height $n+1$ is part of a weak solution. Let us take $s \in A$ such that $h(s) = n + 1$. Since all successors of s have height less or equal to n, we know that they are all part of a weak solution. If one of them is part of weak *Max*-solution Σ, then $\Sigma \cup \{s\}$ is a weak *Max*-solution that contains s. Otherwise, each successor s' is part of a weak *Min*-solution $\Sigma_{s'}$. Since weak *Min*-solutions are closed under union, we can take the union of these *Min*-solutions and obtain a *Min*-solution: $\Sigma = \bigcup_{s \to s'} \Sigma_{s'}$. It is easy to see that $\Sigma \cup \{s\}$ is a weak *Min*-solution that contains s.

The same idea works if we take $s \in B$ instead, and we omit the details. □

Definition 4. A *strong solution* to a two-outcome game is a partition of S, $(\Sigma, S \setminus \Sigma)$ such that

$$\text{If } s \in F \text{ then } s \in \Sigma \Leftrightarrow \mathit{Win} \in \lambda(s) \tag{2.7}$$

$$\text{If } s \in A \text{ then } s \in \Sigma \Leftrightarrow \exists s \xrightarrow{\mathit{Max}} s', s' \in \Sigma \tag{2.8}$$

$$\text{If } s \in B \text{ then } s \in \Sigma \Leftrightarrow \forall s \xrightarrow{\mathit{Min}} s', s' \in \Sigma \tag{2.9}$$

Proposition 1 and Proposition 2 directly lead to the following caracterisation of strong solutions.

Theorem 1 (Existence and unicity of a strong solution)**.** *If the game graph is a finite DAG, then a unique strong solution exists and can be constructed by taking Σ*

to be the states that are part of some weak Max*-solution and $S \setminus \Sigma$ to be the states that are part of some weak* Min*-solution.*

Remark 3. *A strong solution is a pair of weak solutions that are maximal for the inclusion relation.*

From now on, we will extend the notion of *won* and *lost* states to non-final states by saying that a state is *won* if it is part of a weak *Max*-solution and that it is *lost* if it is part a weak *Min*-solution.

It is now possible to give a formal meaning to Allis's notion of solving a game ultra-weakly, weakly, and strongly [3, 156].

Remark 4. *A game with a specified initial state s_0 is* ultra-weakly solved *when we have determined whether s_0 was won or s_0 was lost.*

A game with a specified initial state s_0 is weakly solved *when we have exhibited a weak solution that contains s_0.*

A game is strongly solved *when we have exhibited a strong solution.*

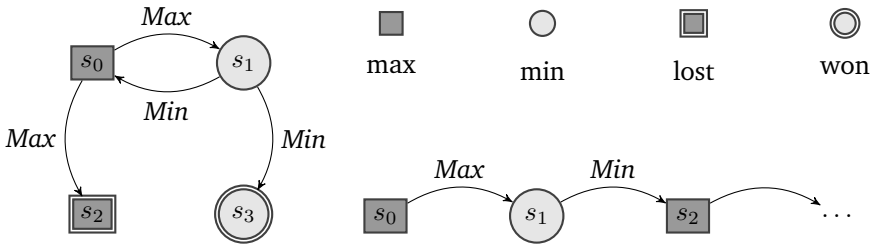
While the finite DAG assumption in the previous statements might seem quite restrictive, it is the simplest hypothesis that leads to well-definedness and exclusion of won and lost values for non terminal states. Indeed, if the game graph allows cycles or if is not finite, then Theorem 1 might not hold anymore.

Example 1. Consider the game $G_1 = \langle S_1, R, \rightarrow_1, L, \lambda_1 \rangle$ with 4 states, $S_1 = \{s_1, s_2\}$, and a cyclic transition relation $\rightarrow_1$. The transition relation is defined extensively as $s_0 \xrightarrow{Max}_1 s_1$, $s_0 \xrightarrow{Max}_1 s_2$, $s_1 \xrightarrow{Min}_1 s_0$, and $s_1 \xrightarrow{Min}_1 s_3$. The only final state to be labelled *Win* is s_3. A graphical representation of G_1 is presented in Figure 2.1a.

G_1 admits two strong solutions, $(\{s_0, s_1, s_3\}, \{s_2\})$ and $(\{s_3\}, \{s_0, s_1, s_2\})$. While s_3 is undeniably a win state and s_2 is undeniably a lost state, s_0 and s_1 could be considered both at the same time.

Example 2. Consider the game $G_2 = \langle S_2, R, \rightarrow_2, L, \lambda_2 \rangle$ defined so that there are infinitely many states, $S_2 = \{s_i, i \in \mathbb{N}\}$, and the transition relation $\rightarrow_2$ is such that $s_{2i} \xrightarrow{Max} s_{2i+1}$ and $s_{2i+1} \xrightarrow{Min} s_{2i+2}$. λ_2 is set so that no states are labelled *Win*. A graphical representation of G_2 is presented in Figure 2.1b.

G_2 admits two strong solutions, $(S_2, \emptyset)$ and $(\emptyset, S_2)$. Put another way, we can consider that all the states are winning or that all the states are losing.



(a) Cyclic game graph (b) Infinite game graph

Figure 2.1: Examples of two-outcome games in which the conclusions of Theorem 1 do not apply.

In practice, the vast majority of games actually satisfy this hypothesis. Take CHESS, for instance, while it is usually possible from a state s to reach after a couple moves a state s' where the pieces are set in the same way as in s, s and s' are actually different. If the sequence of moves that lead from s to s' is repeated in s' we reach yet another state s'' with the same piece setting. However, s'' is a final state because of the *threefold repetition* rule whereas s and s' were internal states. As a consequence, s' is a different state than s since the aformentionned sequence of moves does not have the same effect. Therefore, in such a modeling of CHESS, the game graph is acyclic. The 50-moves rule, acyclicity, and the fact that there finitely many piece settings ensure that there are finitely many different states.

Another modeling of CHESS only encodes the piece setting into the state and relies on the history of reached position to determine values for position. While introducing dependency on the history is not necessary to define CHESS and gives rise to a complicated model in which very few formal results have been established, it is popular among game programmers and researchers as it allows a representation with fewer distinct states.

The ancient game of GO takes an alternative approach to deal with short loops in the piece setting graph. The *Ko* rule makes it illegal to play a move in a position s that would lead to the same piece setting as the predecessor of s. This rule makes it necessary to take into account short term history. Observe that the Ko rule does not prevent cycles of length greater than two in the piece

setting graph. Another rule called *superko* rule makes such cycles illegal, but the superko rule has only been adopted by Chinese, America, and New Zealand GO federation. On the other hand, Japanese and Korean rules allow long cycles in the piece setting graph. As a consequence, the game can theoretically last for an infinite number of moves without ever reaching a final position. In practice, when a long cycle occurs in Japanese and Korean professionnal play, the two players can agree to stop the game. This is not understood as a draw as it would be in CHESS, but is rather seen as a *no result* outcome and the players are required to play a new game to determie a winner.

In the rest of this Chapter, we will mostly be concerned with providing search algorithms for weakly solving games that have a finite game graph with a DAG structure.

2.2 Depth First Search

In this Section, we present a simple game search algorithm to weakly solve two-outcome games called Depth First Search (DFS). It is a direct adaptation of the graph search algorithm of the same name. Indeed, DFS performs a depth-first traversal of (an unfolding of) the game graph until the set of visited nodes contains a weak solution for the initial state. Pseudo-code for DFS is presented in Algorithm 1.

Algorithm 1: Pseudo-code for the DFS algorithm.

```
dfs(state s)
  switch on the turn of s do
    case s ∈ F
      return Win ∈ λ(s)
    case s ∈ A
      foreach s' in {s', s --Max--> s'} do
        if dfs(s') then return true
      return false
    case s ∈ B
      foreach s' in {s', s --Min--> s'} do
        if not dfs(s') then return false
      return true
```

Remark 5. *The specification of DFS in Algorithm 1 is non-deterministic. One consequence of this non-determinism is that the algorithm might or might not converge for a given game. This will be expanded upon in Example 3.*

If the game graph is a finite DAG, then its unfolding is a finite tree. The DFS algorithm visits each state of the unfolded tree at most once so it can only visit a finite number of states in the unfolding. This is summed up in Proposition 3.

Proposition 3 (Termination)**.** *If the game graph is a finite DAG, then the DFS algorithm terminates.*

Proof. If the game graph is a finite DAG, then the height of states is well-defined. We prove the proposition by induction on the height of the argument state s.

Base case: $h(s) = 0$. When s is a final state, DFS returns without performing a recursive call.

Induction case: Assume DFS terminates whenever given an argument of height less or equal to n and prove that it terminates when given an argument s of height $n + 1$. Let s be a state of height $n + 1$. s is either a max state or a min state, and all its successors have height less or equal to n. Since the game graph is finite, we know that s only has finitely many successors. We conclude that when DFS is called with s as argument, finitely many recursive calls to DFS are performed and so the algorithm terminates. □

The DFS algorithm is correct, that is, if `dfs(`s`)` terminates, then it returns `true` only when there exists a weak *Max*-solution containing s, and it returns `false` only when there exists a weak *Min* solution containing s.

Proposition 4 (Correctness)**.** *If DFS returns* `true` *when given argument* s*, then there exists a weak* Max*-solution including* s*. If it returns* `false` *then there is a weak* Min*-solution including* s*.*

Proof. Induction on the depth of the call-graph of DFS. □

This property establised by Proposition 4 does not rely on the finite DAG assumption. However, if we make the finite DAG assumption, then Proposition 3 and 4 combine and lead to the following completeness result.

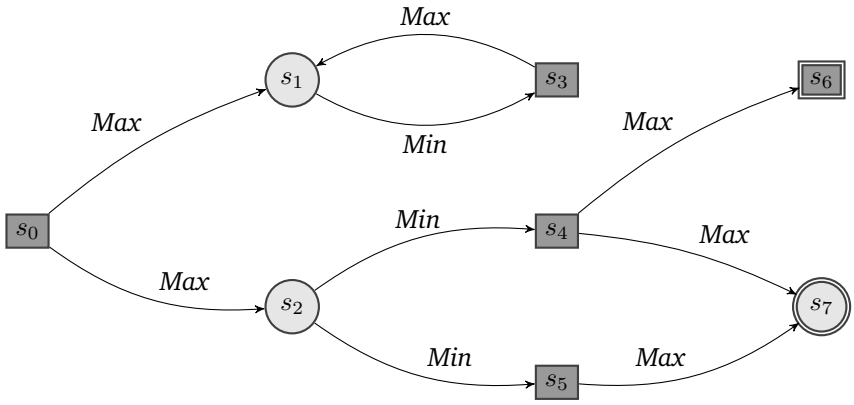


Figure 2.2: Example of a two-outcome game in which the DFS algorithm might or might not terminate. s_6 is a lost final node and s_7 is a won final node.

Theorem 2 (Completeness in finite DAG)**.** *When called on a state s of a game which graph is a finite DAG, then DFS terminates and returns* `true` *exactly when s is won and returns* `false` *exactly when s is lost.*

If the game graph is allowed to be infinite or to contain cycles, then DFS might or might not terminate.

Example 3. Consider the game presented in Figure 2.2. The graph of the game presents a cycle, $\{s_1, s_3\}$, and the game indeed has two strong solutions, $(\{s_0, s_1, s_2, s_3, s_4, s_5, s_7\}, \{s_6\})$ and $(\{s_0, s_2, s_4, s_5, s_7\}, \{s_1, s_3, s_6\})$. s_0 is part of the *Max* weak-solution of every strong solution, so it can be considered as a won state. Still, it is possible that a call to DFS with s_0 as argument does not terminate and it is also possible that it terminates and returns `true`. Indeed, s_0 has two successor states, s_1 and s_2, and DFS is called recursively in either of the two non-deterministically. On the one hand, if the first recursive call from s_0 takes s_1 as argument, then the algorithm gets stuck in an infinite loop. On the other hand, if the first recursive call from s_0 takes s_2 as argument, then that calls returns `true` and the loop is shortcut without calling DFS on s_1.

2.3 Best First Search

We propose in this section a generic Best First Search (BFS) framework. The framework can be seen as a template that makes it easy to define game tree search algorithms for two-outcome games. The framework is general enough to encompass PNS and MCTS in particular.

2.3.1 Formal definitions

Definition 5. An *information scheme* is a tuple $\langle V, \top, \bot, \preccurlyeq, H \rangle$ such that V is a set of *information values*; $\top \subset V$ and $\bot \subset V$ are two distinguished sets of *top values* and *bottom values*.

$\preccurlyeq$ is a selection relation parameterized by a player and an information context: $\forall v \in V$ we have $\preccurlyeq^v_{\text{max}}$ and $\preccurlyeq^v_{\text{min}}$ two total preorders on V.

H is an update function parameterized by a player. It aggregates multiple pieces of information into a single information value. Since we allow pieces of information to be repeated, we need to use multisets rather than sets. We have $H_{\text{max}} : \mathbb{N}^V \to V$ and $H_{\text{min}} : \mathbb{N}^V \to V$.

This set represents the information that can be associated to nodes of the tree. The intended interpretation of $v_1 \preccurlyeq^v_p v_2$ is that v_2 is preferred to v_1 by player p under context v.

We extend the notation for the selection relation as follows: $v_1 \preccurlyeq_p v_2$ is short for $\forall v \in V, v_1 \preccurlyeq^v_p v_2$. It is not hard to see that $\preccurlyeq_p$ is also a total preorder.

Definition 6. We define the set of *solved values* as $S = \top \cup \bot$ and the set of *unsolved values* as $U = V \setminus S$.

Example 4. Let the set of information values be the real numbers with both infinites: $V = \mathbb{R} \cup \{-\infty, +\infty\}$, the bottom values be the singleton $\bot = \{-\infty\}$, and the top values be $\top = \{+\infty\}$. We can define a selection relation $\preccurlyeq$ that is independent of the context as $\forall x \in V$, $a \preccurlyeq^x_{Max} b$ iff $a \leq b$ and $a \preccurlyeq^x_{Min} b$ iff $a \geq b$. Finally, we can take for the update function the standard $\max$ and $\min$ operators: $H_{Max} = \max$ and $H_{Min} = \min$. Together, these elements make an information scheme: MinMaxIS $\stackrel{\text{def}}{=} \langle V, \top, \bot, \preccurlyeq, H \rangle$.

The set of solved values is $S = \{-\infty, +\infty\}$ and the set of unsolved values is $U = \mathbb{R}$.

Definition 7. An information scheme $\langle V, \top, \bot, \preccurlyeq, H \rangle$ is *well formed* if the following requirements are met. The top and bottom values do not overlap.

$$\top \cap \bot = \emptyset \tag{2.10}$$

The selection relation avoids lost values for max and avoid won values for min.

$$\bot \prec_{\max} V \setminus \bot \text{ and } \top \prec_{\min} V \setminus \top \tag{2.11}$$

A top value is sufficient to allow a top max update. A multiset with only bottom values leads to a bottom max update.

$$\begin{aligned} M^* \cap \top \neq \emptyset \text{ implies } H_{\max}(M) \in \top \\ M^* \subseteq \bot \text{ implies } H_{\max}(M) \in \bot \end{aligned} \tag{2.12}$$

A bottom value is sufficient to allow a bottom min update. A multiset with only top values leads to a top min update.

$$\begin{aligned} M^* \cap \bot \neq \emptyset \text{ implies } H_{\min}(M) \in \bot \\ M^* \subseteq \top \text{ implies } H_{\min}(M) \in \top \end{aligned} \tag{2.13}$$

An update cannot create top and bottom values without justification.

$$M^* \cap S = \emptyset \text{ implies } H_p(M) \notin S \tag{2.14}$$

Proposition 5. *The information scheme defined in Example 4 is well formed.*

We will only be interested in well-formed information scheme.

Definition 8. Let $G = \langle S, R, \rightarrow, L, \lambda \rangle$ be a two-outcome game, and let $I = \langle V, \top, \bot, \preccurlyeq, H \rangle$ be a well-formed information scheme, and ζ be an information function $\zeta : S \rightarrow V$. Then $\langle G, I, \zeta \rangle$ is a *best first scheme* if the following two constraints are met.

- The information function needs to be *consistent*. If a state s is associated to a top value $\zeta(s) \in \top$ then there exists a weak *Max*-solution containing s. Conversely, if a state s is associated to a bottom value $\zeta(s) \in \bot$ then there exists a weak *Min*-solution containing s.

- The information function needs to be *informative*. If a state is final, then it is associated to a solved value by the information function. $s \in F \Rightarrow \zeta(s) \in S$.

While the consistency requirement might seem daunting at first, there are multiple ways to create information function that are consistent by construction. For instance, any function returning a top or a bottom value when and only when the argument state is final is consistent.

2.3.2 Algorithmic description

We now show how we can construct a best first search algorithm based on a best first scheme as defined in Definition 8. The basic idea is to progressively build a tree in memory and to associate an information value and a game state to each node of the tree until a weak solution can be derived from the tree.

We assume that each node n of the tree gives access to the following fields. n.info $\in V$ is the information value associated to the node. n.state $\in S$ is the state associated to the node. If n is not a leaf, then n.chidren is the set of children of n. If n is not the root node, then n.parent is the parent node of n. We allow comparing nodes directly based on the selection relation $\preccurlyeq$: for any two nodes n_1 and n_2, we have $n_1 \preccurlyeq_p^v n_2$ iff n_1.info $\preccurlyeq_p^v n_2$.info. We also indulge in applying the update function to nodes rather than to the corresponding information value: if C is a set of nodes and M is the corresponding multiset of information values, $M = \{n.\text{info}, n \in C\}$, then $H(C)$ is short for $H(M)$.

Algorithm 2 develops an exploration tree for a given state s. To be able to orient the search efficiently towards proving a win or a loss for player *Max* instead of just exploring, we need to attach additional information to the nodes beyond their state label.

If the root node is not solved, then more information needs to be added to the tree. Therefore a (non-terminal) leaf needs to be expanded. To select it, the tree is recursively descended selecting at each node the next child according to the $\preccurlyeq$ relation.

Once the node to be expanded, n, is reached, each of its children are added to the tree and they are evaluated with ζ. Thus the status of n changes from leaf to internal node and its value has to be updated with the H function. This update may in turn lead to an update of the value of its ancestors.

After the values of the nodes along the descent path are updated, another leaf can be expanded. The process continues iteratively with a descent of the tree, its expansion and the consecutive update until the root node is solved.

Algorithm 2: Generic pseudo-code for a best-first search algorithm in two-player games.

```
extend(node n)
  foreach s' in {s', n.state → s'} do
      new node n'
      n'.state ← s' ; n'.info ← ζ(s')
      Add n' to n.children

backpropagate(node n)
  old_info ← n.info
  switch on the turn of n.state do
      case n.state ∈ A  n.info ← H_max(n.children)
      case n.state ∈ B  n.info ← H_min(n.children)
  if old_info = n.info ∨ n = r then return n
  else return backpropagate(n.parent)

bfs(state s)
  new node r
  r.state ← s ; r.info ← ζ(s)
  n ← r
  while r.info ∉ S do
      while n is not a leaf do
          C ← n.children
          switch on the turn of n.state do
              case n.state ∈ A  n ← any element ∈ C maximizing ≼_max^n.info
              case n.state ∈ B  n ← any element ∈ C maximizing ≼_min^n.info
      extend(n)
      n ← backpropagate(n)
  return r
```

2.3.3 Properties

We turn on to proving a few properties of BFS algorithms generated with the proposed framework. That is, we assume given a blabla and we prove formal properties on this system. Thus any *best first scheme* constructed with this framework will satisfy the properties presented in this section. The typical application of this work is to alleviate the proof burden of the algorithm designer as it is now sufficient to show that a new system is actually a best first scheme.

Proposition 6 (Correctness)**.** *If n.info $\in \top$ then n.state is contained in a weak* Max*-solution. Conversely if n.info $\in \bot$ then n.state is contained in a weak* Min*-solution.*

Proof. Structural induction on the current tree using the consistency of the evaluation function. □

Proposition 7. *If n is a node reached by the BFS algorithm during the descent, then it is not solved yet: n.info $\notin S$.*

Proof. Proof by induction.

Base case: When n is the root of the tree, $n = r$, we have r.info $\notin S$ by hypothesis.

Induction case: assume n is a child node of p, p.info $\notin S$, and n maximizes the selection relation. Let $M = \{n'.\text{info for } n' \in p.\text{children}\}$. If p is a *Max* state, p.state $\in A$, we note that p.info $= H_{\max}(M)$.

Given that p is not solved, we have in particular that p.info $\notin \bot$ and therefore $M^* \not\subseteq \bot$ from Equation (2.12). As a result, at least one element in M does not belong to $\bot$. Let n' be a node such that n'.info $\notin \bot$. n maximizes $\preccurlyeq_{\max}$ so n' is not strictly preferred to n. Since $\preccurlyeq_{\max}$ avoids lost values and n'.info is not lost, then we know that n cannot be lost either (Equation (2.11)). Thus, n.info $\notin \bot$.

We also have that p.info $\notin \top$ and therefore $M^* \cap \top = \emptyset$ from Equation (2.12). As a result, no element in M belongs to $\top$. In particular, n.info $\notin \top$ and so we conclude that n is not be solved: n.info $\notin S$.

The case where p is a *Min* state is similar and is omitted. □

Proposition 8 (Progression)**.** *If n is a leaf node reached by the BFS algorithm during the descent, then the corresponding position is not final: n.state $\notin F$.*

Proof. We have assumed in Definition 8 that the evaluation function ζ was informative. That is, n.state $\in F$ implies n.info $\in S$. We know from Proposition 7 that n.info $\notin S$. Hence, we can conclude that n.state $\notin F$. □

The direct consequence of Proposition 8 is that the `extend()` procedure always add at least one node to the tree. Therefore, the size of the tree grows after each iteration.

Proposition 9 (Convergence in finite games)**.** *If the game graph is finite and acyclic, the* BFS *algorithm terminates.*

We will see in Section 2.4, 2.5, and 2.6 a few classical algorithms can be expressed the suggested formalism and inherit its theoretical properties. Many more are possible, for instance the results we obtain also apply to the best first scheme derived from Example 4 .

2.4 Proof Number Search

PNS [4, 74] is a best first search algorithm that enables to dynamically focus the search on the parts of the search tree that seem to be easier to solve. PNS based algorithms have been successfully used in many games and especially as a solver for difficult games such as CHECKERS [137], SHOGI [141], and GO [73].

There has been a lot of developments of the original PNS algorithm [4]. An important problem related to PNS is memory consumption as the tree has to be kept in memory. In order to alleviate this problem, V. Allis proposed PN^2 [3]. It consists in using a secondary PNS at the leaves of the principal PNS. It allows to have much more information than the original PNS for equivalent memory, but costs more computation time. PN^2 has recently been used to solve FANORONA [131].

The main alternative to PN^2 is the DFPN algorithm [103]. DFPN is a depth-first variant of PNS based on the iterative deepening idea. DFPN will explore the game tree in the same order as PNS with a lower memory footprint but at the cost of re-expanding some nodes.

We call *effort numbers* heuristic numbers which try to quantify the amount of information needed to prove some fact about the value of a position. The higher the number, the larger the missing piece of information needed to prove

the result. When an effort number reaches 0, then the corresponding fact has been proved to be true, while if it reaches ∞ then the corresponding fact has been proved to be false.

In PNS we try to decide whether a node belongs to a weak *Max*-solution. That is, we simultaneously try to find a weak *Max*-solution containing it and to prove that it does not belongs to any weak *Max*-solution. We will use the standard PNS terminology for the remaining of this Section, that is, we say that we *prove a node* when we find a weak *Max*-solution containing it, and that we *disprove a node* when we find a weak *Min*-solution containing it.

2.4.1 The Proof Number Search Best First Scheme

We use $\mathbb{N}^*$ to denote the set of positive integers: $\mathbb{N}^* = \{1, 2, \ldots\}$.

The information value associated to nodes contains two parts: we have $v = (p, d)$ with $p, d \in \mathbb{N} \cup \{\infty\}$. The *proof number* ($p$) represents an estimation of the remaining effort needed to prove the node, while the *disproof number* (d) represents an estimation of the remaining effort needed to disprove the node. When *Max* solution has been found we have $p(n) = 0$ and $d(n) = \infty$, and when a *Min* solution has been found we have $p(n) = \infty$ and $d(n) = 0$.

$$\begin{aligned} V &= \mathbb{N}^* \times \mathbb{N}^* \cup \{(0, \infty), (\infty, 0)\} \\ \top &= \{(0, \infty)\} \text{ and } \bot = \{(\infty, 0)\} \end{aligned} \tag{2.15}$$

The basic idea in PNS is to strive for proofs that seem to be easier to obtain. Thus, we define the selection relation so that if *Max* is on turn, then the selected child minimizes the proof number and if *Min* is on turn, the selected child minimizes the disproof number.

$$\begin{aligned} (p, d) &\preccurlyeq_{\max} (p', d') \text{ iff } p' \leq p \\ (p, d) &\preccurlyeq_{\min} (p', d') \text{ iff } d' \leq d \end{aligned} \tag{2.16}$$

If an internal node n corresponds to a *Max* position, then proving one child of n is sufficient to prove n and disproving n requires disproving all its children. As a consequence, the (greatest lower bound on the) amount of effort needed to prove n is the amount of effort needed for the easiest child of n to be proved, and the amount of effort needed to prove n is bounded below by the sum of

efforts for all children of n. A similar intuition for *Min* nodes leads to the update functions.

$$\begin{aligned} H_{\max}(M) &= \left(\min_{(p,d)\in M} p, \sum_{(p,d)\in M} d \right) \\ H_{\min}(M) &= \left(\sum_{(p,d)\in M} p, \min_{(p,d)\in M} d \right) \end{aligned} \tag{2.17}$$

It is not hard to see that for any multiset of V, M, we have $H_{Max}(M) \in V$ and $H_{Min}(M) \in V$. Therefore we have an information scheme.

Proposition 10. *The information scheme is well-formed.*

The evaluation function, also known as the *initialization function* in the PNS litterature, is a simple admissible bound on the effort to prove or disprove a node. If the node corresponds to a final position, then we know its value. If it is a *Win*, the remaining effort need to prove it is $p = 0$ and the remaining effort to disprove it can be set to $d = \infty$ since we know this node cannot ever be disproved. Conversely, if the final node is not a *Win* then we set the proof number to ∞ and the disproof number to 0.

If the node n corresponds to a non-final position, then (dis)proving will require expanding at least one node (this very node n), so we set $p = 1$ and $d = 1$. This can be summed up with the following initialization formulas.

$$\begin{aligned} \forall s \in A \cup B, &\zeta(s) = (1,1) \\ \forall s \in F, \mathit{Win} \in \lambda(s) &\Rightarrow \zeta(s) = (0,\infty) \\ \forall s \in F, \mathit{Win} \notin \lambda(s) &\Rightarrow \zeta(s) = (\infty,0) \end{aligned} \tag{2.18}$$

ζ is consistent and informative. Therefore, we have a best first scheme. From Proposition 6 and Proposition 9 we have that PNS is correct, and converges in finite acyclic games.

Example 5. Here is an example of finite game with a cycle in which PNS does not converge.

Example 6. Here is an example of an infinite game without cycles in which PNS does not converge.

2.5 Monte Carlo Tree Search

MCTS is a very successful algorithm for multiple complete information games such as GO [40, 41, 54, 86, 114], HEX [25, 7], or LINES OF ACTION [166].

MCTS is a recent game tree search technique based on multi-armed bandit problems [20]. MCTS has enabled a huge leap forward in the playing level of artificial GO players. It has been extended to prove wins and losses under the name MCTS Solver [165, 46]. It is this MCTS Solver algorithm that we describe here.

The basic idea in MCTS is to evaluate whether a state s is favourable to *Max* via Monte Carlo playouts in the tree below s. A Monte Carlo playout is a random path of the tree below s ending in a terminal state. Performing a playout and checking the type of the corresponding terminal state can be done as demonstrated in Algorithm 3.

Monte Carlo programs usually deal with transpositions the *simple way*: they do not modify the UCT formula and develop a DAG instead of a tree.

Algorithm 3: Pseudo-code for a Monte Carlo Playout.

```
playout(state s)
  switch on the turn of s do
    case s ∈ F ∧ Win ∈ λ(s)  return 1
    case s ∈ F ∧ Win ∉ λ(s)  return 0
    otherwise
      s' ← random state such that s → s'
      return playout(s')
```

MCTS explores the Game Automaton (GA) in a best first way by using aggregates of information given by the playouts.

$$\begin{aligned} V &= \mathbb{N} \times \mathbb{N}^* \times \{0,1,2\} \\ \top &= \mathbb{N} \times \mathbb{N}^* \times \{2\} \\ \bot &= \mathbb{N} \times \mathbb{N}^* \times \{0\} \end{aligned} \tag{2.19}$$

An information value is a triple (r,t,s) where t denotes the total number of playouts rooted below n and r denotes the number of playouts ending in a *Win* state among them.

We also have the label s that represents the solution status and allows to avoid solved subtrees. s can take three values: 0, 2, or 1. These values respectively mean that the corresponding node was weakly *Min*-solved, weakly *Max*-solved, or not solved yet for this node.

$$\begin{aligned}
(r,t,s) \preccurlyeq_{\max}^{(r_0,t_0,s_0)} (r',t',s') \text{ iff } &\begin{cases} s < s' \\ s = s' \text{ and } \frac{r}{t} + \sqrt{\frac{2\ln t_0}{t}} \leq \frac{p'}{t'} + \sqrt{\frac{2\ln t_0}{t'}} \end{cases} \\
(r,t,s) \preccurlyeq_{\min}^{(r_0,t_0,s_0)} (r',t',s') \text{ iff } &\begin{cases} s' < s \\ s = s' \text{ and } \frac{-r}{t} + \sqrt{\frac{2\ln t_0}{t}} \leq \frac{-r'}{t'} + \sqrt{\frac{2\ln t_0}{t'}} \end{cases}
\end{aligned} \tag{2.20}$$

When a node is not solved yet, we are faced with an exploration-exploitation dilemma between running playouts in nodes which have not been explored much (t is small) and running playouts in nodes which seem successful (high $\frac{r}{t}$ ratio). This concern is addressed using the UCB formula [9, 20].

$$\begin{aligned}
&\forall s \in A \cup B, \zeta(s) = (\texttt{playout}(s), 1, 1) \\
&\forall s \in F, \textit{Win} \in \lambda(s) \Rightarrow \zeta(s) = (1,1,2) \\
&\forall s \in F, \textit{Win} \notin \lambda(s) \Rightarrow \zeta(s) = (0,1,0)
\end{aligned} \tag{2.21}$$

To initialize a value corresponding to a non terminal position s we call the `playout(`s`)` procedure (Algorithm 3). If the position s is terminal, then the initial value depends on whether s is a *Win* state.

$$\begin{aligned}
H_{\max}(M) &= (\sum_{(r,t,s)\in M} r, \sum_{(r,t,s)\in M} t, \max_{(r,t,s)\in M} s) \\
H_{\min}(M) &= (\sum_{(r,t,s)\in M} r, \sum_{(r,t,s)\in M} t, \min_{(r,t,s)\in M} s)
\end{aligned} \tag{2.22}$$

The total number of playouts rooted at a node can be viewed as the sum of the number of playouts rooted at each child. Similarly, the number of playouts ending in a *Win* state is the sum of the corresponding number at each child.

Proposition 11. *ζ is informative and consistent, the information scheme is well-formed, and so we have a best first scheme.*

Remark 6. *The evaluation of a leaf node in MCTS as presented in Equation 2.21 takes the form of games played randomly until a terminal position. It can also be the value of a heuristical evaluation function after a few random moves [92, 166]. We denote the latter variant as MCTS-E.*

2.6 Product Propagation

PP is a way to backup probabilistic information in a two-player game tree search [144]. It has been advocated as an alternative to minimaxing that does not exhibit the minimax pathology [110, 69, 70].

PP was recently proposed as an algorithm to solve games, combining ideas from PNS and probabilistic reasoning [146]. In Stern's paper, PP was found to be about as performant as PNS for capturing GO problems.

In this Chapter, we express PP as an instance of the BFS framework presented in Section 2.3 and conduct an extensive experimental study of PP, comparing it to various other paradigmatic solving algorithms and improving its memory consumption and its solving time. Doing so, we hope to establish that PP is an important algorithm for solving games that the game search practician should know about. Indeed, we exhibit multiple domains in which PP performs better than the other tested game solving algorithms.

The baseline game tree search algorithms that we use to establish PP's value are DFS (see Section 2.2); PNS [4, 155, 74] (see Section 2.4); and MCTS Solver [165] which was recently used to solve the game of HAVANNAH on size 4 [46] (see Section 2.5).

In PP, each node n is associated to a single number PP called the probability propagation number for n, such that $\mathrm{PP} \in [0, 1]$. The PP of a leaf corresponding to a *Max* win is 1 and the PP of a *Max* loss is 0.

$$V = [0, 1],\ \top = \{1\},\ \text{and}\ \bot = \{0\} \tag{2.23}$$

The probability propagation number of a node n can intuitively be understood as the likelihood of n being a *Max* win given the partially explored game tree. With this interpretation in mind, natural update rules can be proposed.

If n is an internal *Min* node, then it is a win for *Max* if and only if all children are win for *Max* themselves. Thus, the probability that n is win is the joint probability that all children are win. If we assume all children are independent, then we obtain that the PP of n is the product of the PP of the children for *Min* nodes. A similar line of reasoning leads to the formula for *Max* nodes.

$$\begin{aligned} H_{\max}(M) &= 1 - \prod_{p \in M} (1-p) \\ H_{\min}(M) &= \prod_{p \in M} p \end{aligned} \tag{2.24}$$

To define the PP of a non-terminal leaf l, the simplest is to assume no information is available and initiate the PP information value to $\frac{1}{2}$.

$$\begin{aligned} &\forall s \in A \cup B, \zeta(s) = 0.5 \\ &\forall s \in F, \mathit{Win} \in \lambda(s) \Rightarrow \zeta(s) = 1 \\ &\forall s \in F, \mathit{Win} \notin \lambda(s) \Rightarrow \zeta(s) = 0 \end{aligned} \tag{2.25}$$

Note that this explanation is just a loose interpretation of the probability propagation numbers and not a formal justification. Indeed, the independence assumption does not hold in practice, and in concrete games n is either a win or a loss for *Max* but it is not a random event. However, this probabilistic analogy mainly serves as an intuition for the algorithm and it is reasonable not to feel constrained by the lack of independance as the algorithm performs well nontheless.

To be able to use the generic BFS framework, we still need to specify which leaf of the tree is to be expanded. The most straightforward approach is to select the child maximizing PP when at a *Max* node, and to select the child minimizing PP when at a *Min* node.

$$\begin{aligned} p \preccurlyeq_{\max} p' &\text{ if and only if } p \leq p' \\ p \preccurlyeq_{\min} p' &\text{ if and only if } p' \leq p \end{aligned} \tag{2.26}$$

Proposition 12. *ζ is informative and consistent, the information scheme is well-formed, and so we have a best first scheme.*

Note that it is also possible to use a heuristical evaluation function taking values in $(0, 1)$ to evaluate leaves.

2.6.1 Experimental Results

While the performance of PP as a solver has matched that of PNS in GO [146], it has proven to be disappointing in SHOGI.[1] We now exhibit several domains where the PP search paradigm outperforms more classical algorithms.

In the following sets of experiments, we do not use any domain specific knowledge besides an evaluation function where appropriate. We are aware that the use of such techniques would improve the solving ability of all our programs. Nevertheless, we believe that showing that a generic and non-optimized implementation of PP performs better than generic and non-optimized implementations of PNS, MCTS, or DFS in a variety of domains provides good reason to think that the ideas underlying PP are of importance in game solving.

Besides PP, PNS, MCTS, and DFS, we also try to incorporate transpositions in PP and PNS [139]. We thus obtain PP with Transpositions (PPT) and PNS with Transpositions (PNT). Finally, we also adapt the PN^2 [19] idea to PP and try the resulting PP^2 algorithm, that is, instead of directly using an heuristical evaluation function to evaluate leaves that correspond to internal position, we perform a nested call to PP.

2.6.2 Results on the game of Y

The game of Y was discovered independently by Claude Shannon in the 50s, and in 1970 by Schensted and Titus. It is played on a triangular board with a hexagonal paving. Players take turns adding one stone of their color on empty cells of the board. A player wins when they succeed in connecting all three edges with a single connected group of stones of their color. Just as HEX, Y enjoys the no-draw property.

The current best evaluation function for Y is the *reduction evaluation function* [162]. This evaluation function naturally takes values in $[0, 1]$ with 0 (resp. 1) corresponding to a *Min* (resp. *Max*) win.

PNS with the mobility initialization could not solve any position in less than 3 minutes in a preliminary set of about 50 positions. As a result we did not include this solver in our experiment with a larger set of positions. The experiments on Y was carried out as follows. We generated $77,012$ opening positions on a board

[1] Akihiro Kishimoto, personnal communication.

Table 2.1: Number of positions solved by each algorithm and number of positions on which each algorithm was performing best.

	PP	MCTS	MCTS-E
Positions solved	77,010	76,434	69,298
Solved fastest	68,477	3,645	4,878
Fewest iterations	22,621	35,444	18,942

of size 6. We then ran PP using the reduction evaluation function, MCTS using playouts with a random policy, and a variant of MCTS using the same reduction evaluation instead of random playouts (MCTS-E). For each solver, we recorded the total number of positions solved within 60 seconds. Then, for each solving algorithm, we computed the number of positions among those $77,012$ which were solved faster by this solver than by the two other solver, as well as the number of positions which needed fewer iterations of the algorithm to be solved. The results are presented in Table 2.1.

We see that the PP algorithms was able to solve the highest number of positions, $77,010$ positions out of $77,012$ could be solved within 60 seconds. We also note that for a very large proportion of positions ($68,477$), PP is the fastest algorithm. However, MCTS needs fewer iterations than the other two algorithms on $35,444$ positions. A possible interpretation of these results is that although iterations of MCTS are a bit more informative than iterations of PP, they take much longer. As a result, PP is better suited to situations where time is the most important constraint, while MCTS is more appropriate when memory efficiency is a bottleneck. Note that if we discard MCTS-E results, then $72,830$ positions are solved fastest by PP, $4,180$ positions are solved fastest by MCTS, $30,719$ positions need fewest iterations to be solved by PP, and $46,291$ need fewest iterations by MCTS.

Figure 2.3 displays some of these results graphically. We sampled about 150 positions of various difficulty from the set of $77,012$ Y positions, and plotted the time needed to solve such positions by each algorithm against the time needed by PP. We see that positions that are easy for PP are likely to be easy for both MCTS solvers, while positions hard for PP are likely to be hard for both other solvers as well.

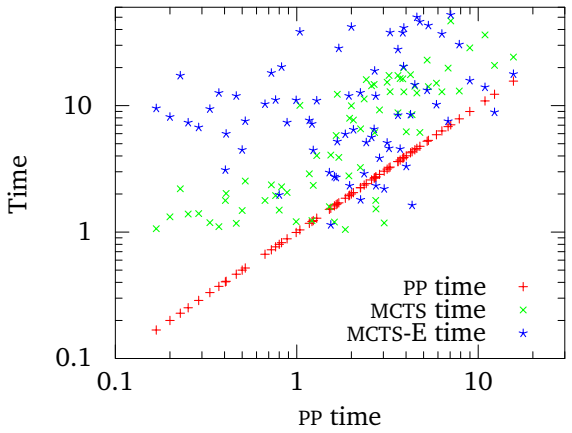


Figure 2.3: Time needed to solve various opening positions in the game of Y.

2.6.3 Results on DOMINEERING

DOMINEERING is played on a rectangular board. The first player places a vertical 2×1 rectangle anywhere on the board. The second player places an horizontal 2×1 rectangle, and the games continues like that until a player has no legal moves. The first player that has no legal moves has lost.

DOMINEERING has already been studied in previous work by game search specialists as well as combinatorial game theorists [18, 82].[2] While these papers focusing on DOMINEERING obtain solution for relatively large boards, we have kept ourselves to a naive implementation of both the game rules and the algorithms. In particular, we do not perform any symmetry detection nor make use of combinatorial game theory techniques such as decomposition into subgames.

We presents results for the following algorithms: DFS, PNT [139], PN^2 [19], PP, PPT and PP^2. The PNS algorithm could not find a single solution within 10^7 node expansion when transpositions where not detected and it is thus left out.

For PP variants, the probability of a non solved leaf is computed as the

[2]Some results can also be found on `http://www.personeel.unimaas.nl/uiterwijk/Domineering_results.html`.

Table 2.2: Number of node expansions needed to solve various sizes of DOMINEERING.

	7×6	6×6	5×6
DFS	$6,387,283,988$	$38,907,049$	$701,559$
PNT	$> 10^7$	$> 10^7$	$1,002,277$
PN^2	$> 511,568$	$> 154,107$	$17,236$
PP	$> 10^7$	$5,312,292$	$836,133$
PPT	$> 10^7$	$419,248$	$140,536$
PP^2	$1,219,024$	$29,870$	$9,986$

Table 2.3: Time (s) needed to solve various sizes of DOMINEERING.

	7×6	6×6	5×6
DFS	$5,656$	40.68	0.87
PNT			5.92
PN^2	$> 153,000$	$> 10,660$	78.7
PP		19.79	2.9
PPT		4.12	1.02
PP^2	$4,763$	21.53	2.15

number of legal moves for the vertical player divided by the sum of the number of legal moves for each player. For PNS variants the mobility heuristic is used to compute the proof numbers and the disproof numbers at non solved leaves.

Tables 2.2 and 2.3 give the number of nodes and times for different algorithms solving DOMINEERING. DFS is turned into the alpha-beta algorithm and is enhanced with transposition tables, killer moves, the history heuristic and an evaluation function. We can see that on the smallest 5×6 board alpha-beta gives the best results. On the larger 6×6 board PPT becomes the best algorithm by far. On the largest 7×6 board, most of the algorithms run out of memory, and the best algorithm is now PP^2 that outperforms both alpha-beta and PN^2.

In their paper, Breuker et al. have shown that the use of transposition tables and symmetries increased significantly the performance of their alpha-beta (that is, DFS) implementation [18]. While, our proof-of-concept implementation does not take advantage of symmetries, our results show that transpositions are of great importance in the PP paradigm as well.

Table 2.4: Number of node expansions needed to solve various sizes of NOGO.

	18×1	20×1	22×1
DFS	$4,444,384$	$154,006,001$	$3,133,818,285$
PNT	$2,015,179$	$> 10^7$	$> 10^7$
PN^2	$> 22,679$	$> 29,098$	
PP	$1,675,159$	$> 10^7$	$> 10^7$
PPT	$206,172$	$657,045$	$4,853,527$
PP^2	$14,246$		

Table 2.5: Time (s) needed to solve various sizes of NOGO.

	18×1	20×1	22×1
DFS	10.43	361.0	$7,564$
PNT	144.2	> 809	
PN^2	$> 3,607$	$> 4,583$	
PP	39.96	> 257.0	
PPT	21.06	85.11	801.0
PP^2	109.7		

2.6.4 Results on NOGO

NOGO is the misere version of the game of GO. It was presented in the BIRS 2011 workshop on combinatorial game theory [28].[3] The first player to capture has lost.

We present results for the following algorithms: DFS, PNT [139], PN^2 [19], PP, PPT and PP^2. Again, the PNS algorithm could not find a single solution within 10^7 node expansion and is left out.

For standard board sizes such as 4×4 or 5×4, DFS gives the best results among the algorithms we study in this paper. We have noticed that for $N \times 1$ boards for $N > 20$, PPT becomes competitive. Results for a few board sizes are given in Table 2.4 for the number of nodes and in Table 2.5 for the times.

[3] http://www.birs.ca/events/2011/5-day-workshops/11w5073

2.6.5 Conclusion

In this Section, we have presented how to define and use Product Propagation (PP) in order to solve abstract two-player games. We briefly described how to extend PP so as to handle transpositions and to reduce memory consumption with the PP^2 algorithm. For the three games that have been tested (i.e., Y, DOMINEERING, and NOGO), we found that our extensions of PP are able to better solve games than the other solving algorithms.

Being a BFS algorithm, PP is quite related to PNS and MCTS. As such, it seems natural to try and adapt ideas that proved successful for these algorithms to the Product Propagation paradigm. For instance, while PNS and PP are originally designed for two-outcome games, future work could adapt the ideas underlying MOPNS [123] (see Section 3.7) to turn PP into an algorithm addressing more general games. Adapting more elaborate schemes for transpositions could also prove interesting [100, 73, 125].

3 Multi-Outcome Games

We define a formal model of deterministic two-player perfect information zero-sum games called multi-outcome game*. We adapt the concept of information scheme to multi-outcome game and obtain a Best First Search (BFS) framework.*

We show that a generalization of Monte Carlo Tree Search (MCTS) Solver, termed Score Bounded Monte Carlo Tree Search (SBMCTS), can be obtained as an instance of the BFS framework. We then develop a principled approach to create a multi-outcome information scheme based on two-outcome information scheme that we call multization*. We use it to derive a new Multiple-Outcome Proof Number Search (MOPNS) algorithm that generalizes Proof Number Search (PNS) to multi-outcome games.*

The Chapter includes results from the following papers.

[26] Tristan Cazenave and Abdallah Saffidine. Score bounded Monte-Carlo tree search. In H. van den Herik, Hiroyuki Iida, and Aske Plaat, editors, Computers and Games*, volume 6515 of* Lecture Notes in Computer Science*, pages 93–104. Springer-Verlag, Berlin / Heidelberg, 2011. ISBN 978-3-642-17927-3. doi: 10.1007/978-3-642-17928-0_9*

[123] Abdallah Saffidine and Tristan Cazenave. Multiple-outcome proof number search. In Luc De Raedt, Christian Bessiere, Didier Dubois, Patrick Doherty, Paolo Frasconi, Fredrik Heintz, and Peter Lucas, editors, 20th European Conference on Artificial Intelligence (ECAI)*, volume 242 of* Frontiers in Artificial Intelligence and Applications*, pages 708–713, Montpellier, France, August 2012. IOS Press.*

ISBN 978-1-61499-097-0. doi: 10.3233/978-1-61499-098-7-708

Contents

3.1 Introduction

Many interesting games have more than two outcomes, for instance CHESS, DRAUGHTS and CONNECT FOUR have three outcomes: *Win*, *Draw*, and *Lose*. A game of WOODPUSH of size s has a number of possible outcomes bounded by $s \times s \times (s+1)$. We describe the game of WOODPUSH in Section 3.8.2. Matches in General Game Playing (GGP) typically are associated to an integer score in $[0, 100]$. For many games, it is not only interesting to know whether the maximizing player can obtain the maximal outcome, but also what is the exact score of the game. That is, what is the best outcome the maximizing player can achieve assuming perfect play from the opponent.

3.2 Model

Definition 9. A *multi-outcome game* is a transition system $\langle S, R, \rightarrow, L, \lambda \rangle$ where the following restriction holds.

- There are two distinguished agents *Max* $\in R$ and *Min* $\in R$
- State turns are exclusive: $\neg\exists s_1, s_2, s_3 \in S, s_1 \xrightarrow{Max} s_2 \wedge s_1 \xrightarrow{Min} s_3$.
- There is a finite ordered set of distinguished labels called *outcomes*: $\mathbb{O} = \{o_1 < o_2 < \cdots < o_{m-1}\} \subseteq L$;

We define A and B as the sets of states that allow respectively *Max* and *Min* transitions: $A = \{s \in S, \exists s' \in S, s \xrightarrow{Max} s'\}$ and $B = \{s \in S, \exists s' \in S, s \xrightarrow{Min} s'\}$.

We say that a state is *final* if there it allows no transition for *Max* nor *Min*. We denote the set of final states by F. $F = S \setminus (A \cup B)$. States that are not final are called *internal*. For two states $s_1, s_2 \in S$, we say that s_2 is a *successor* of s_1 if it can be reached by a *Max* or a *Min* transition. Formally, we write $s_1 \rightarrow s_2$ when $s_1 \xrightarrow{Max} s_2 \vee s_1 \xrightarrow{Min} s_2$.

From the turn exclusivity assumption, we derive that A, B, and F constitute a partition of S.

Let o_0 and o_m two new state labels not appearing in L. We denote $\mathbb{O} \cup \{o_0, o_m\}$ with $\overline{\mathbb{O}}$. We extend the ordering on $\mathbb{O}$ to $\overline{\mathbb{O}}$ by taking $o_0 < o_i < o_m$ for all $0 < i < m$.

Definition 10. The *score* of a final state $s \in F$, $\sigma(s)$, is defined as the maximum outcome if any outcome appears in s, and o_0 otherwise. $\sigma(s) = \max(\mathbb{O} \cap \lambda(s))$ if $\mathbb{O} \cap \lambda(s) \neq \emptyset$, and $\sigma(s) = o_0$ if $\mathbb{O} \cap \lambda(s) = \emptyset$.

Definition 11. A *weak* Max-*o-solution* to a multi-outcome game is a labelling of states $\Sigma^o_{\max} \subseteq S$ such that

- If $s \in F$ then $s \in \Sigma^o_{\max} \Rightarrow \sigma(s) \geq o$
- If $s \in A$ then $s \in \Sigma^o_{\max} \Rightarrow \exists s \xrightarrow{Max} s', s' \in \Sigma^o_{\max}$
- If $s \in B$ then $s \in \Sigma^o_{\max} \Rightarrow \forall s \xrightarrow{Min} s', s' \in \Sigma^o_{\max}$

Definition 12. A *weak* Min-o*-solution* to a multi-outcome game is a labelling of states $\Sigma^o_{\text{min}} \subseteq S$ such that

- If $s \in F$ then $s \in \Sigma^o_{\text{min}} \Rightarrow \sigma(s) < o$
- If $s \in A$ then $s \in \Sigma^o_{\text{min}} \Rightarrow \forall s \xrightarrow{Max} s', s' \in \Sigma^o_{\text{min}}$
- If $s \in B$ then $s \in \Sigma^o_{\text{min}} \Rightarrow \exists s \xrightarrow{Min} s', s' \in \Sigma^o_{\text{min}}$

Definition 13. A *weak-solution* to a multi-outcome game is a pair of labellings of states $(\Sigma^{o_i}_{\text{max}}, \Sigma^{o_{i+1}}_{\text{min}}) \subseteq S \times S$ such that $\Sigma^{o_i}_{\text{max}}$ is a weak *Max*-o_i-solution and $\Sigma^{o_{i+1}}_{\text{min}}$ is a weak-*Min*-o_{i+1}-solution, and with non-empty intersection: $\Sigma^{o_i}_{\text{max}} \cap \Sigma^{o_{i+1}}_{\text{min}} \neq \emptyset$. In that case, for any state s in the intersection we say that o_i is the value of s.

Conversely, it is possible to prove that if the game graph is finite Direct Acyclic Graph (DAG), then each state is associated to exactly one value.

We say that a multi-outcome game with a distinguished initial state s_0 is weakly-solved when we can exhibit a weak-solution containing s. Multi-outcome games that have been weakly solved include CONNECT 4 [2], CHECKERS [137], and FANORONA [132].

3.3 Iterative perspective

Let $\langle S, R, \rightarrow, L, \lambda \rangle$ with outcome set $\mathbb{O}$ a multi-outcome game. For any outcome $o \in \mathbb{O}$, $\langle S, R, \rightarrow, L, \lambda \rangle$ can be seen as a two-outcome game with distinguished label o. The transformed games have exactly the same rules and game graph as the original one but have different distinguished outcomes.

Proposition 13. *We can combine solutions on the various two-outcome games and obtain a solution to the multi-outcome game.*

If there are more than two possible outcomes, the minimax value of the starting position can still be found with a two-outcome algorithm by using a binary search on the possible outcomes [4]. If there are m different outcomes, then the binary search will make about $\lg(m)$ calls to the two-outcome algorithm. If the score of a position is already known, e.g., from expert knowledge, but needs to be proved, then two calls to a two-outcome algorithm are necessary and sufficient.

3.4 MiniMax and Alpha-Beta

The MiniMax value of a game tree is calculated based on the assumption that the two players, called *Max* and *Min*, will choose their next move such that when it is *Max*'s turn he will select the action that maximizes his gain while *Min* will select the one that minimizes it on his turn. MiniMax values are propagated from the leaves of the game tree to its root using this rule. Alpha-beta uses the MiniMax value to prune a subtree when it has proof that a move will not affect the decision at the root node [118]. This happens when a partial search of the subtree reveals that the opponent has the opportunity to lower an already established MiniMax value backed up from a different subtree.

Algorithm 4: Pseudo-code for the MiniMax algorithm.

```
minimax(state s)
  switch on the turn of s do
    case s ∈ F
      return σ(s)
    case s ∈ A
      α ← o_0
      foreach s' in {s', s --Max--> s'} do
        α ← max{α, minimax(s')}
      return α
    case s ∈ B
      β ← o_m
      foreach s' in {s', s --Min--> s'} do
        β ← min{β, minimax(s')}
      return β
```

3.5 Multiple-Outcome Best First Search

We have seen in Section 3.3 that it was possible to use two-outcome algorithms iteratively to solve multi-outcome games. While this approach works in principle and was sometimes used in games with three outcomes [131], it seems wasteful not to reuse the state-space exploration effort between the different passes of

Algorithm 5: Pseudo-code for the alpha-beta algorithm.

```
alpha-beta(state s, outcome α, outcome β)
  switch on the turn of s do
    case s ∈ F
      return σ(s)
    case s ∈ A
      foreach s' in {s', s --Max--> s'} do
        α ← max{α, alpha-beta(s', α, β)}
        if β ≤ α then break
      return α
    case s ∈ B
      foreach s' in {s', s --Min--> s'} do
        β ← min{β, alpha-beta(s', α, β)}
        if β ≤ α then break
      return β
```

the search. In this Section we propose a one pass Multiple-Outcome Best First Search (MOBFS) algorithm that can solve multi-outcome games.

3.5.1 Formal Definitions

Definition 14. An *information scheme* is a tuple $\langle V, \mathbb{O}, \top, \bot, \preccurlyeq, H \rangle$ such that

- V is a set of information values. This set represents the information that can be associated to nodes of the tree.
- $\top = \{\top^o\}_{o \in \overline{\mathbb{O}}}$ and $\bot = \{\bot^o\}_{o \in \overline{\mathbb{O}}}$ are two collections of distinguished set of values, where for all $o \in \mathbb{O}$, $\top^o \subset V$ and $\bot^o \subset V$. We call $\top^o$ the set of *positive values* associated to o and $\bot^o$ the set of *negative values* associated to o.
- $\preccurlyeq$ is a selection relation parameterized by a player and a context based on a pair of information values. $\forall v, v' \in V$ we have $\preccurlyeq_{\max}^{v,v'}$ and $\preccurlyeq_{\min}^{v,v'}$ two total preorders on V. The intended interpretation of $v_1 \preccurlyeq_p^{v,v'} v_2$ is that v_2 is preferred to v_1 by player p under context (v, v').
- H is an update function parameterized by a player. It aggregates multiple pieces of information into a single information value. Since we allow

pieces of information to be repeated, we need to use multisets rather than sets. $H_{\max} : \mathbb{N}^V \to V$ and $H_{\min} : \mathbb{N}^V \to V$.

The intended interpretation of $\top$ and $\bot$ is that if a value belongs to $\top^o$ then we know that *Max* can ensure an outcome o or better is reached. Conversely, if a value belongs to $\bot^o$ then we know that *Min* can ensure an outcome o or better is not reached.

Definition 15. We define the set of *solved values* as $S = \bigcup_{0 \leq i < m} \bot^{o_{i+1}} \cap \top^{o_i}$ and the set of *unsolved values* as $U = V \setminus S$.

As an example of an information scheme, we propose the following *Blind* information scheme. While the definition is very straightforward and is not based on elaborate concepts, we will see later (in Proposition 15 that this information scheme is precise enough to allow solving multi-outcome games. Our presentation follows Definition 14.

Example 7. Let $\mathbb{O} = \{o_1, \ldots, o_{m-1}\}$ and let *Blind* be the information scheme defined by

$$\begin{aligned} V &= \{(p,n), 0 \leq p < n \leq m\} \\ \top^{o_i} &= \{(p,n) \in V, i \leq p\} \\ \bot^{o_i} &= \{(p,n) \in V, n \leq i\} \end{aligned} \tag{3.1}$$

Intuitively the first field of the information value reflects the highest outcome that has been proved to be achievable by *Max*. The second field reflects the lowest outcome known not to be achievable by *Max*.

$$\begin{aligned} (p,n) &\preccurlyeq_{\max} (p',n') \quad \text{iff } n \leq n' \\ (p,n) &\preccurlyeq_{\min} (p',n') \quad \text{iff } p' \leq p \end{aligned} \tag{3.2}$$

The selection relation can be seen as *Max* being optimistic and always preferring values with a better potential outcome. Conversely, *Min* prefers values

$$\begin{aligned} H_{\max}(M) &= (\max_{(p,n)\in M} p, \max_{(p,n)\in M} n) \\ H_{\min}(M) &= (\min_{(p,n)\in M} p, \min_{(p,n)\in M} n) \end{aligned} \tag{3.3}$$

The set of solved values for Blind is S such that

$$S = \bigcup_{0 \leq i < m} \{(p, n), i \leq p < n \leq i + 1\} = \{(i, i + 1), 0 \leq i \leq m\} \tag{3.4}$$

Definition 16. An information scheme $\langle V, \mathbb{O}, \top, \bot, \preccurlyeq, H \rangle$ is *well formed* if the following requirements are met.

- The sets of positive and negative values are respectively decreasing and increasing. For all $o_i < o_j$, $\top^{o_j} \subseteq \top^{o_i}$ and $\bot^{o_i} \subseteq \bot^{o_j}$.
- Any value is positive for o_0 and any value is negative for o_m. $\top^{o_0} = V$ and $\bot^{o_m} = V$.
- No value is both positive and negative for a given outcome o. That is, the corresponding sets do not overlap $\top^o \cap \bot^o = \emptyset$.
- The selection relation avoids dominated values: $\top^{o_i} \cap \bot^{o_{i+1}} \prec_{\max} V \setminus (S \cup \bot^{o_{i+1}})$ and $\top^{o_i} \cap \bot^{o_{i+1}} \prec_{\min} V \setminus (S \cup \top^{o_i})$.
- A positive value is sufficient to allow a positive max update: $M^* \cap \top^{o_i} \neq \emptyset$ implies $H_{\max}(M) \in \top^{o_i}$. A multiset with only negative values leads to a negative max update: $M^* \subseteq \bot^{o_i}$ implies $H_{\max}(M) \in \bot^{o_i}$.
- A negative value is sufficient to allow a negative min update: $M^* \cap \bot^{o_i} \neq \emptyset$ implies $H_{\min}(M) \in \bot^{o_i}$. A multiset with only positive values leads to a positive min update: $M^* \subseteq \top^{o_i}$ implies $H_{\min}(M) \in \top^{o_i}$.
- An update cannot create positive and negative values without justification. For any $o_i \in \overline{\mathbb{O}}$, $M^* \cap (\top^{o_i}) = \emptyset$ implies $H_p(M) \notin \top^{o_i}$ and $M^* \cap (\bot^{o_i}) = \emptyset$ implies $H_p(M) \notin \bot^{o_i}$.

As a consequence we have $\top^{o_m} = \bot^{o_0} = \emptyset$.

We can practice proving well-formedness on this simple information scheme presented in Example 7. As we shall see in Section 3.5.2, knowing that an information scheme is well-formed allows to derive many useful properties such as correctness of the resulting BFS algorithm.

Proposition 14. *The Blind information scheme presented in Example 7 is well-formed.*

Proof. The positive and negative sets are decreasing and increasing. For all $o_i < o_j$,

$$\begin{aligned} \top^{o_j} &= \{(p,n), i \leq j \leq p < n \leq m\} \subseteq \top^{o_i} = \{(p,n), i \leq p < n \leq m\} \\ \bot^{o_i} &= \{(p,n), 0 \leq p < n \leq i \leq j\} \subseteq \bot^{o_j} = \{(p,n), 0 \leq p < n \leq j\} \end{aligned} \tag{3.5}$$

The positive set for o_0 and the negative set for o_m are exactly the possible information values.

$$\top^{o_0} = \bot^{o_m} = \{(p,n), 0 \leq p < n \leq m\} = V \tag{3.6}$$

The top and bottom values for outcome o_i do not overlap.

$$\top^{o_i} \cap \bot^{o_i} = \{(p,n), i \leq p < n \leq i\} = \emptyset \tag{3.7}$$

- The selection relation for *Max* avoids dominated values. On the one hand $\top^{o_i} \cap \bot^{o_{i+1}} = \{(p,n), i \leq p < n \leq i+1\} = \{(i, i+1)\}$. On the other hand, $V \setminus (S \cup \bot^{o_{i+1}}) \subseteq V \setminus \bot^{o_{i+1}} = \{(p,n), i+1 < n\}$, and we have indeed $(i, i+1) \prec_{\max} \{(p,n), i+1 < n\}$.

  The same reasoning shows that the selection relation for *Min* avoid dominated values.

- Let M be a multiset of information values, and let $(p_0, n_0) = H_{\max}(M)$. Assume $M^* \cap \top^{o_i} \neq \emptyset$ and take $(p,n) \in M^* \cap \top^{o_i}$. We know that $i \leq p$, and also that $p \leq p_0$. Therefore $i \leq p_0$ and $H_{\max}(M) \in \top^{o_i}$.

  Similarly, if we assume $M^* \subseteq \bot^{o_i}$, then for all $(p,n) \in M^*$, we have $n \leq i$. As a result, $n_0 \leq i$ and $H_{\max}(M) \in \bot^{o_i}$.

- The same reasoning on the *Min* update function leads to the expected result.

- Finally, it is easy to derive a similar argument to show that an update cannot create positive and negative values without justification.

□

Definition 17. Let $G = \langle S, R, \to, L, \lambda \rangle$ be a multi-outcome game with outcome set $\mathbb{O}$, $I = \langle V, \mathbb{O}, \top, \bot, \preccurlyeq, H \rangle$ be a well-formed information scheme, and ζ be an information function $\zeta : S \to V$. Then $\langle G, I, \zeta \rangle$ is a *best first scheme* if the following constraints are met.

- The information function needs to be *consistent*. If a state s is associated to a top value $\zeta(s) \in \top^{o_i}$ then there exists a weak *Max*-o_i-solution containing s. Conversely, if a state s is associated to a bottom value $\zeta(s) \in \bot^{o_{i+1}}$ then there exists a weak *Min*-o_{i+1}-solution containing s.
- The evaluation function needs to be *informative*. If a state is final, then it is associated to a solved value by the evaluation function. $s \in F \Rightarrow \zeta(s) \in S$.

Proposition 15. *Consider ζ such that for every final state $s \in F$, $\zeta(s) = (i, i+1)$ where $o_i = \sigma(s)$, and for every non final state $s \in A \cup B$, $\zeta(s) = (0, m)$. Then combining ζ to the Blind information scheme defined in Example 7 gives a best first scheme.*

3.5.2 Properties

We define the score of a node as the score of the corresponding position: $\sigma(n) = \sigma(n.\text{state})$. We define the pessimistic and optimistic bounds for an information value v as $\text{pess}(v) = \max\{o \in \overline{\mathbb{O}}, v \in \bot^o\}$ and $\text{opti}(v) = \min\{o \in \overline{\mathbb{O}}, v \in \top^o\}$. The definition of these bounds is naturally extended to nodes.

Definition 18. The *pessimistic* bound for a node n is defined as $\text{pess}(n) = \max\{o \in \overline{\mathbb{O}}, n..\text{info} \in \bot^o\}$. Similarly, the *optimistic* bound for a node n is defined as $\text{opti}(n) = \min\{o \in \overline{\mathbb{O}}, n..\text{info} \in \top^o\}$.

The pessimistic bound is the worst value possible for n consistent with the current information in the tree. And the optimistic bound is the best value possible for n consistent with the current information in the tree. It can be useful in some implementations or proofs to observe that the bounds can be computed recursively from the leaf nodes upwards.

Proposition 16. *Let n be an internal node.*

$$\textit{If } n.\text{state} \in A \begin{cases} \text{pess}(n) = \max\limits_{c \in n.\text{children}} \text{pess}(c) \\ \text{opti}(n) = \max\limits_{c \in n.\text{children}} \text{opti}(c) \end{cases} \tag{3.8}$$

$$\textit{If } n.\text{state} \in B \begin{cases} \text{pess}(n) = \min\limits_{c \in n.\text{children}} \text{pess}(c) \\ \text{opti}(n) = \min\limits_{c \in n.\text{children}} \text{opti}(c) \end{cases} \tag{3.9}$$

Proof. Proof by induction on the height of the node making use of the well-formedness of the heredity function H. □

The following inequality gives their name to the bounds.

Proposition 17. *The pessimistic (resp. optimistic) bound of a node is a lower (resp. upper) bound on the score associated to the corresponding position.*

$$\mathrm{pess}(n) \leq \sigma(n) \leq \mathrm{opti}(n). \tag{3.10}$$

Proof. Proof by induction on the height of the node making use of the consistency of the evaluation function ζ. □

For any node n, we know the exact score of the position corresponding to n as soon as the two bounds match $\mathrm{pess}(n) = \mathrm{opti}(n)$. Although the definition is different, these bounds coincide with those described in SBMCTS [26].

We also define *relevancy bounds* that are similar to alpha and beta bounds in the classic Alpha-Beta algorithm [118]. For a node n, the *lower* relevancy bound is noted $\alpha(n)$ and the *upper* relevancy bound is noted $\beta(n)$. These bounds are calculated using the optimistic and pessimistic bounds as follows. If n is the root of the tree, then $\alpha(n) = \mathrm{pess}(n)$ and $\beta(n) = \mathrm{opti}(n)$. Otherwise, n has a parent f in the tree. In that case, we use the relevancy bounds of the parent of n: if $n \in f$.children, we set $\alpha(n) = \max\{\alpha(f), \mathrm{pess}(n)\}$ and $\beta(n) = \min\{\beta(f), \mathrm{opti}(n)\}$.

The relevancy bounds of a node n take their name from the fact that if $\sigma(n) \leq \alpha(n)$ or if $\sigma(n) \geq \beta(n)$, then having more information about $\sigma(n.\text{state})$ will not contribute to solving the root of the tree. Therefore they enable safe pruning.

Proposition 18. *For each node n, if we have $\beta(n) \leq \alpha(n)$ then the the subtree of n need not be explored any further.*

Subtrees starting at a pruned node can be completely removed from the main memory as they will not be used anymore in the proof. This improvement is crucial as lack of memory is one of the main bottleneck of PNS and MOPNS.

3.5.3 Score Bounded Monte Carlo Tree Search

An MCTS solver which backs up exact MiniMax values of the sequential zero-sum two-outcome game *Lines of Action* was introduced in [165]. SBMCTS [26] expands on this idea and generalized the MCTS solver concept to any sequential zero-sum game. Score bounded search allows for pruning in the absence of exact MiniMax values as long as there is some information available to establish bounds.

Because simulations do not usually methodically explore the game tree, it is to be expected that we cannot easily assign MiniMax values to the states when we explore them as we are only sampling the subtree below. Even though we may not have explored every reachable state, the sampling information builds up and can be used to get tighter and tighter bounds on state values. These bounds are called pessimistic and optimistic, referring to the payoff *Max* believes he can get in the worst and best case, respectively. The default bounds are the minimum and maximum achievable values. Instead of backing up a MiniMax value, the bounds of a state are deduced from the bounds of subsequent states and used in Alpha-Beta fashion by checking whether lower and upper bounds coincide.

An information value is a 4-tuple $\boldsymbol{v} = (r, t, p, n)$.[1] Let n be a node in the BFS tree and $\boldsymbol{v}$ the associated information value. v_1 denotes the total reward accumulated from playouts rooted below node n, v_2 denotes the total number of such playouts. v_3 is a the greatest lower bound on the score of n that has been obtained so far while v_4 is the smallest upper bound on the score of n.

$$\begin{aligned} V &= \{(r,t,p,n), r \in \mathbb{N}, t \in \mathbb{N}^*, 0 \leq p < n \leq m\} \\ \top^{o_i} &= \{\boldsymbol{v} \in V, i \leq v_3\} \\ \bot^{o_i} &= \{\boldsymbol{v} \in V, v_4 \leq i\} \end{aligned} \tag{3.11}$$

The selection relation relies on the Upper Confidence Bound (UCB) formula to decide which node is more interesting unless the score bounds prove that one is inferior or superior to the other. The only contextual information that we need to compare two sibling nodes is the number of playouts accumulated

[1] We extend the vector notation to tuples: if $\boldsymbol{v} = (r, t, p, n)$ then $v_1 = r$, $v_2 = t$, $v_3 = p$, and $v_4 = n$.

below the father. This contextual information is needed for the computation of the exploration factor in the UCB formula. We will use $\preccurlyeq^t$ as short for $\preccurlyeq^{(r,t,p,n)}_{(r',t',p',n')}$.

$$\boldsymbol{v} \preccurlyeq^{t_0}_{\max} \boldsymbol{v'} \text{ iff } \begin{cases} v_4 \leq v'_3 \text{ or} \\ v'_3 < v_4,\ v_3 < v'_4, \text{ and } \frac{v_1}{v_2(m-1)} + \sqrt{\frac{2\ln t_0}{v_2}} \leq \frac{v'_1}{v'_2(m-1)} + \sqrt{\frac{2\ln t_0}{v'_2}} \end{cases}$$
$$\boldsymbol{v} \preccurlyeq^{t_0}_{\min} \boldsymbol{v'} \text{ iff } \begin{cases} v'_4 \leq v_3 \text{ or} \\ v'_3 < v_4,\ v_3 < v'_4, \text{ and } \frac{-v_1}{v_2(m-1)} + \sqrt{\frac{2\ln t_0}{v_2}} \leq \frac{-v'_1}{v'_2(m-1)} + \sqrt{\frac{2\ln t_0}{v'_2}} \end{cases} \tag{3.12}$$

The accumulated reward is the sum of the accumulated rewards over the children nodes and the total number of playouts is also the sum of the number of playouts over the cildren. The score bounds are the greastest or smallest bounds found among the bounds of the children depending on which player controls the node.

$$\begin{aligned} H_{\max}(M) &= \left(\sum_{\boldsymbol{v}\in M} v_1, \sum_{\boldsymbol{v}\in M} v_2, \max_{\boldsymbol{v}\in M} v_3, \max_{\boldsymbol{v}\in M} v_4 \right) \\ H_{\min}(M) &= \left(\sum_{\boldsymbol{v}\in M} v_1, \sum_{\boldsymbol{v}\in M} v_2, \min_{\boldsymbol{v}\in M} v_3, \min_{\boldsymbol{v}\in M} v_4 \right) \end{aligned} \tag{3.13}$$

To initialize a value corresponding to a non terminal position s we call the `playout(`s`)` procedure (Algorithm 6). As no definite information is known about the game theoretic value associated to s, the score bounds are set to safe initial values. If the position s is terminal, then the information value depends on the score of s, $\sigma(s)$.

$$\begin{aligned} &\forall s \in A \cup B, \zeta(s) = (\texttt{playout}(s), 1, 0, m) \\ &\forall s \in F, \zeta(s) = (i, 1, i, i+1) \text{ such that } o_i = \sigma(s) \end{aligned} \tag{3.14}$$

The set of solved values for SBMCTS is S such that

$$S = \{(r, t, i, i+1), r \in \mathbb{N}, t \in \mathbb{N}^*, 0 \leq i < m\} \tag{3.15}$$

$$S = \{\boldsymbol{v} \in V, v_4 = v_3 + 1\} \tag{3.16}$$

Algorithm 6: Pseudo-code for a Monte Carlo Playout in SBMCTS.

```
playout(state s)
  if s ∈ F then return i such that o_i = σ(s)
  else
      s' ← random state such that s → s'
      return playout(s')
```

Theorem 3. *The information scheme for SBMCTS is well-formed.*

3.6 Multization

MOBFS is a new framework to derive a BFS algorithm for multiple-outcome games based on a BFS algorithm for two-outcome game. We apply the MOBFS idea to PNS and Product Propagation (PP) to create MOPNS (Section 3.7).

We have seen in Section 2.3 that a two-outcome BFS algorithm could be defined by specifying a *two-outcome best first scheme*. We now show how such a scheme can be used to build a multi-outcome best first scheme. Create such a multi-outcome information scheme only requires a base two-outcome information scheme and a priority relation π.

Assume the outcome set is $\mathbb{O} = \{o_1 < o_2 < \cdots < o_{m-1}\}$. The multi-outcome BFS algorithm we propose will associate $m-1$ node values $v_1, \ldots, v_{m-1}$ to each node in the constructed tree. For a node n, value $v_i(n)$ corresponds to the current information about the decision problem corresponding to o_i:

> Is the game theoretic value of n greater or equal to o_i?

At each iteration of the new BFS algorithm, we compute which coordinate of the information value at the root maximizes the priority relation and call it *attractive outcome*. We then project the multi-outcome node values according to the attractive outcome and use those projections and the two-outcome BFS algorithm to perform the iteration. Put another way, the attractive outcome is the outcome that constitutes the focus of an iteration of the multi-outcome BFS algorithm.

Definition 19. We say that the priority relation π is *well-formed* if it prefers unsolved values to solved ones: $S \pi U$.

Definition 20. Let $\langle V, \top, \bot, \preccurlyeq, H\rangle$ a two-outcome information scheme, and $\mathbb{O} = \{o_1, \ldots, o_{m-1}\}$ a set of outcomes. We define a multi-outcome information scheme $\langle \boldsymbol{V}, \mathbb{O}, \{\top^o\}_{o\in\overline{\mathbb{O}}}, \{\bot^o\}_{o\in\overline{\mathbb{O}}}, \preccurlyeq, \boldsymbol{H}\rangle$ as follows. The set of information values is the cartesian product of the original set.

$$\boldsymbol{V} = V \times \cdots \times V = V^{m-1} \tag{3.17}$$

For each outcome $o_i \in \overline{\mathbb{O}}$,

$$\begin{aligned} \top^{o_i} &= \{(t_1, \ldots, t_i, v_{i+1}, \ldots, v_{m-1}), t_j \in \top, v_j \in V\} \\ \bot^{o_i} &= \{(v_1, \ldots, v_{i-1}, b_i, \ldots, b_{m-1}), b_j \in \bot, v_j \in V\} \end{aligned} \tag{3.18}$$

with the understanding that $\top^{o_0} = \bot^{o_m} = \{(v_1, \ldots, v_{m-1}), v_j \in V\} = V'$ and $\top^{o_m} = \bot^{o_0} = \emptyset$.

The selection relation uses the priority relation to determine on which coordinate the base selection should be applied.

$$\boldsymbol{v} \preccurlyeq^{\boldsymbol{r},\boldsymbol{u}} \boldsymbol{w} \text{ iff } v_i \preccurlyeq^{r_i,u_i} w_i \text{ where } i \text{ is such that } r_i \text{ maximize } \pi \text{ in } \boldsymbol{r}. \tag{3.19}$$

The update function applies the base update to each coordinate.

$$\boldsymbol{H}(M) = (H(\{v_1\}_{\boldsymbol{v}\in M}), \ldots, H(\{v_{m-1}\}_{\boldsymbol{v}\in M})) \tag{3.20}$$

Theorem 4. *If the base information scheme and the priority relation are well-formed, then the multi-outcome version is well-formed as well.*

We now show that pruning does not interfere with the descent policy in the sense that it will not affect the number of descents performed before the root is solved. For this purpose, we prove that the descent policy does not lead to a node which can be pruned.

Proposition 19. *If r is not solved, then for all nodes n traversed by the root descent policy, $\alpha(n) < o^* \leq \beta(n)$.*

Proof. We first prove the inequality for the root node. If the root position r is not solved, then by definition of the attractive outcome, $o^* > \mathrm{pess}(r) = \alpha(r)$. Using Proposition 20, we know that all outcomes better than the optimistic bound cannot be achieved: $\forall o > \mathrm{opti}(r) = \beta(r)$, $\mathrm{G}(o, r) = \infty$. Since $\mathrm{G}(r, o^*) + \mathrm{S}(r, o^*) \neq \infty$, then $\alpha(r) < o^* \leq \beta(r)$.

For the induction step, suppose n is a *Max* node that satisfies the inequality. We need to show that $c = \arg\min_{c \in \text{chil}(n)} \mathrm{G}(c, o^*)$ also satisfies the inequality. Recall that the pessimistic bounds of n and c satisfy the following order: $\text{pess}(c) \leq \text{pess}(n)$ and obtain the first part of the inequality $\alpha(c) = \alpha(n) < o^*$. From the induction hypothesis, $o^* \leq \beta(n) \leq \text{opti}(n)$, so from Proposition 20 $\mathrm{G}(n, o^*) \neq \infty$, moreover, the selection process ensures that $\mathrm{G}(c, o^*) = \mathrm{G}(n, o^*) \neq \infty$, therefore $\mathrm{G}(c, o^*) \neq \infty$ which using Proposition 21 leads to $o^* \leq \text{opti}(c)$. Thus, $o^* \leq \beta(c)$. The induction step when n is a *Min* node is similar and is omitted. □

3.7 Multiple-Outcome Proof Number Search

In this chapter, we propose a new *effort number* based algorithm that enables to solve games with multiple outcomes. The principle guiding our algorithm is to use the same tree for all possible outcomes. When using a dichotomic PNS, the search trees are independent of each other and the same subtrees are expanded again. We avoid this re-expansion sharing the common nodes. Moreover we can safely prune some nodes using considerations on bounds as in Score Bounded MCTS [26].

MOPNS aims at applying the ideas from PNS to multi-outcome games. However, contrary to dichotomic PNS and iterative PNS, MOPNS dynamically adapts the search depending on the outcomes and searches the same tree for all the possible outcomes.

In PNS, two effort numbers are associated with every node, whereas in MOPNS, if there are m outcomes, then $2m$ effort numbers are associated with every node. In PNS, only completely solved subtrees can be pruned, while pruning plays a more important role in MOPNS and can be compared to alpha-beta pruning.

3.7.1 Effort Numbers

MOPNS also uses the concept of effort numbers but different numbers are used here in order to account for the multiple outcomes. Let n be a node in the game tree, and $o \in \mathbb{O}$ an outcome. The *greater number*, $\mathrm{G}(n, o)$, is an estimation of the number of node expansions required to prove that the value of n is greater than or equal to o (from the point of view of *Max*), while conversely the *smaller*

Outcome	G	S
Win	500	10
Draw	0	∞

Figure 3.1: Example of effort numbers for a three-outcome game with distinguished outcomes $\mathbb{O} = \{\mathit{Win}, \mathit{Draw}\}$

number, $\mathrm{S}(n, o)$, is an estimation of the number of node expansions required to prove that the value of n is strictly smaller than o. If $\mathrm{G}(n, o_i) = \mathrm{S}(n, o_{i+1}) = 0$ then n is solved and its value is $\sigma(n) = o_i$.

Figure 3.1 features an example of effort numbers for a three-outcome game. The effort numbers show that in the position under consideration *Max* can force a draw and it seems unlikely that at that point the *Max* can force a win.

3.7.2 Determination of the effort

The effort numbers of internal nodes are obtained in a very similar fashion to PNS, G is analogous to p and S is analogous to d. Every effort number of a leaf is initialized at 1, while the effort numbers of an internal node are calculated with the sum and min formulae as shown in Figure 3.2a.

If n is a terminal node and its value is $\sigma(n)$, then the effort numbers are associated as shown in Figure 3.2b. We have for all $o \leq \sigma(n)$, $\mathrm{G}(n, o) = 0$ and for all $o \geq \sigma(n)$, $\mathrm{S}(n, o) = 0$.

3.7.3 Properties

$\mathrm{G}(n, o_i) = 0$ (resp. $\mathrm{S}(n, o_{i+1}) = 0$) means that the value of n has been proved to be greater than (resp. smaller) or equal to o_i, i.e., *Max* (resp. *Min*) can force the outcome to be at least o_i (resp. at most o_i). Conversely $\mathrm{G}(n, o_i) = \infty$ means that it is impossible to prove that the value of n is greater than or equal to o_i, i.e., *Max* cannot force the outcome to be greater than or equal to o_i.

As can be observed in Figure 3.1, the effort numbers are monotonic in the outcomes. If $o_i \leq o_j$ then $\mathrm{G}(n, o_i) \leq \mathrm{G}(n, o_j)$ and $\mathrm{S}(n, o_i) \geq \mathrm{S}(n, o_j)$. Intuitively, this property states that the better an outcome is, the harder it will be to obtain it or to obtain better.

Node type	$\mathrm{G}(n,o)$	$\mathrm{S}(n,o)$
Leaf	1	1
Max	$\min_{c\in\mathrm{chil}(n)} \mathrm{G}(c,o)$	$\sum_{c\in\mathrm{chil}(n)} \mathrm{S}(c,o)$
Min	$\sum_{c\in\mathrm{chil}(n)} \mathrm{G}(c,o)$	$\min_{c\in\mathrm{chil}(n)} \mathrm{S}(c,o)$

(a) Internal node

Outcome	G	S
o_m	∞	0
$\dots$	∞	0
$\sigma(n)$	0	0
$\dots$	0	∞
o_1	0	∞

(b) Terminal node

Figure 3.2: Determination of effort numbers for MOPNS

0 and ∞ are permanent values since when an effort number reached 0 or ∞, its value will not change as the tree grows and more information is available. Several properties link the permanent values of a given node. The proofs are straightforward recursions from the leaves and are omitted for lack of space. Care must only be taken that the initialization of leaves satisfies the property which is the case for all the initializations discussed here.

Proposition 20. *If* $\mathrm{G}(n,o)=0$ *then for all* $o'\leq o$, $\mathrm{S}(n,o')=\infty$ *and similarly if* $\mathrm{S}(n,o)=0$ *then for all* $o'\geq o$, $\mathrm{G}(n,o')=\infty$.

Proposition 21. *If* $\mathrm{G}(n,o)=\infty$ *then* $\mathrm{S}(n,o)=0$ *and similarly if* $\mathrm{S}(n,o)=\infty$ *then* $\mathrm{G}(n,o)=0$.

3.7.4 Descent policy

We call *attracting outcome* of a node n, the outcome $o^*(n)$ that minimizes the sum of the corresponding effort numbers.

$$o^*(n)=\arg\min_o(\mathrm{G}(n,o)+\mathrm{S}(n,o)) \tag{3.21}$$

Put another way, we define the priority relation π as

$$(p, d)\pi(p', d') \text{ if and only if } p + d \leq p' + d' \tag{3.22}$$

As a consequence of the existence of a minimax value for each position, for all node n, there always exists at least one outcome o, such that $\mathrm{G}(n, o) \neq \infty$ and $\mathrm{S}(n, o) \neq \infty$. Hence, $\mathrm{G}(n, o^*(n)) + \mathrm{S}(n, o^*(n)) \neq \infty$.

Consider Figure 3.1, if these effort numbers were associated to a *Max* node, then the attracting outcome would be *Win*, while if they were associated to a *Min* node then the attracting outcome would be *Draw*.

Proposition 22. *For finite two outcome games,* MOPNS *and* PNS *develop the same tree.*

Proof. If we know the game is finite, the *Max* is sure to obtain at least the worst outcome so we can initialize the greater number for the worst outcome to 0, we can also initialize the smaller number for the best outcome to 0. If there are two outcomes only then one is distinguished: $\mathbb{O} = \{\mathit{Win}\}$. We then have the following relation between effort numbers in PNS and MOPNS: $\mathrm{G}(n, \mathit{Win}) = p$, $\mathrm{S}(n, \mathit{Win}) = d$. If the game is finite with two outcomes, then the attracting outcome of the root is *Win*. Hence, MOPNS and PNS behave in the same manner. □

3.7.5 Applicability of classical improvements

Many improvements of PNS are directly applicable to MOPNS. For instance, the *current-node enhancement* presented in [3] takes advantage of the fact that many consecutive descents occur in the same subtree. This optimization allow to obtain a notable speed-up and can be straightforwardly applied to MOPNS.

It is possible to initialize leaves in a more elaborate way than presented in Figure 3.2a. Most initializations available to PNS can be used with MOPNS, for instance the *mobility initialization* [155] in a *Max* node n consists in setting the initial smaller number to the number of legal moves: $\mathrm{G}(n, o) = 1$, $\mathrm{S}(n, o) = |\operatorname{chil}(n)|$. In a *Min* node, we would have $\mathrm{G}(n, o) = |\operatorname{chil}(n)|$, $\mathrm{S}(n, o) = 1$.

A generalization of PN^2 is also straightforward. If n is a new leaf and d descents have been performed in the main tree, then we run a nested MOPNS independent from the main search starting with n as root. After at most d

descents are performed, the nested search is stopped and the effort numbers of the root are used as initialization numbers for n in the main search. We can safely propagate the interest bounds to the nested search to obtain even more pruning.

Similarly, a transformation of MOPNS into a depth-first search is possible as well, adapting the idea of Nagai [103]. Just as in Depth-First Proof Number Search (DFPN), only two threshold numbers would be needed during the descent, one threshold would correspond to the greater number for the current attractive outome at the root and one threshold would correspond to the smaller number for the distractive outcome.

Finally, given that MOPNS is very close in spirit to PNS, a careful implementer should not face many problems adapting the various improvements that make DFPN such a successful technique in practice. Let us mention in particular Nagai's garbage collection technique [103], Kishimoto and Müller's solution to the Graph History Interaction problem [73], and Pawlewicz and Lew's $1 + \varepsilon$ trick [109].

3.8 Experimental results

To assess the validity of our approach, we implemented a prototype of MOPNS and tested it on two games with multiple outcomes, namely CONNECT FOUR and WOODPUSH. Our prototype does not detect transposition and is implemented via the best first search approach described earlier. As such, we compare it to the original best-first variation of PNS, also without transposition detection. Note that the domain of CONNECT FOUR and WOODPUSH are acyclic, so we do not need to use the advanced techniques presented by Kishimoto and Müller to address the Graph History Interaction problem [73]. Additionally, the positions that constitute our testbed were easy enough that they could be solved by search trees of at most a few million nodes. Thus, the individual search trees for PNS as well as MOPNS could fit in memory without ever pruning potentially useful nodes.

In our implementation, the two algorithms share a generic code for the best first search module and only differ on the initialization, the update, and the selection procedures. The experimental results were obtained running OCaml

3.11.2 under Ubuntu on a laptop with Intel T3400 CPU at 2.2 GHz and 1.8 GiB ofmemory.

For each test position and each possible outcome, we performed one run of the PNS algorithm and recorded the time the number of node creation it needed. We then discarded all but the two runs needed to prove the final result. For instance, if a position in WOODPUSH admitted non-zero integer scores between -5 and $+5$ and its perfect play score was 2, we would run PNS ten times, and finally output the measurements for the run proving that the score is greater or equal to 2 and the measurements for the run disproving that the score is greater or equal to 3. This policy is beneficial to PNS compared to doing a binary search for the outcome.

To compare MOPNS to PNS on a wide range of positions, we created the list of all positions reached after a given number of moves from the starting position of a given size. These positions range from being vastly favourable to *Min* to vastly favourable to *Max*, and from trivial (solved in a few milliseconds) to more involved (each run being around two to three minutes).

3.8.1 CONNECT FOUR

CONNECT FOUR is a commercial two-player game where players drop a red or a yellow piece on a 7×6 grid. The first player to align four pieces either horizontally, vertically or diagonally wins the game. The game ends in a draw if the board is filled and neither player has an alignment. The game was solved by James D. Allen and Victor Allis in 1988 [2].

Table 3.1 presents aggregate data over our experiments on size 4×5 and 5×5. In both cases, we used the positions occuring after 4 moves. In the first case, 16 positions among the 256 positions tested were a first player win, 222 were a draw while 18 were a first player loss. In the second list of positions, there were 334 wins, 267 draws, and 24 losses.

Figure 3.3 plots the number of node creations needed to solve each of the 256 4×5 positions. We can see that for a majority of positions, MOPNS needed fewer node creations than PNS. There are 16 positions that needed the same number of node creations by both algorithm and these positions are exactly the positions that are first player wins.

Table 3.1: Cumulated time and number of node creation for the MOPNS and PNS algorithms in the game of CONNECT FOUR. For both algorithm, *Lowest time* indicates the number of positions that were soved faster by this algorithm, while *Lowest node creations* indicates the number of positions which needed fewer node creations.

		MOPNS	PNS
Size 4×5, 256 positions after 4 moves	Total time (seconds)	99	85
	Total node creations	16,947,536	20,175,238
	Lowest time	21	235
	Lowest node creations	227	13
Size 5×5, 625 positions after 4 moves	Total time (seconds)	11,230	9055
	Total node creations	1,557,490,694	1,757,370,222
	Lowest time	55	570
	Lowest node creations	406	140

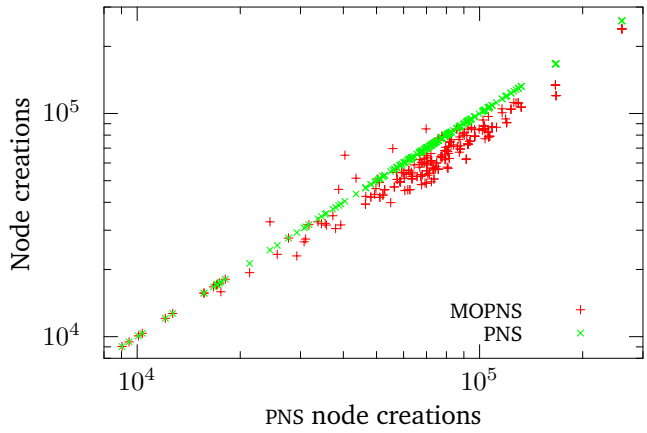


Figure 3.3: Comparison of the number of node creations for MOPNS and PNS for solving 256 CONNECT FOUR positions on size 4×5.

Figure 3.4: WOODPUSH starting position on size $(10, 2)$

3.8.2 WOODPUSH

The game of WOODPUSH is a recent game invented by combinatorial game theorists to analyze a game that involves forbidden repetition of the same position [1, 24]. A starting position consists of some pieces for the left player and some for the right player put on an array of predefined length as shown in Figure 3.4. A Left move consists in sliding one of the left pieces to the right. If some pieces are on the way of the sliding piece, they are jumped over. When a piece has an opponent piece behind it, it can move backward and push all the pieces behind, provided it does not repeat the previous position. The game is won when the opponent has no more pieces on the board. The score of a game is the number of moves that the winner can play before the board is completely empty.

The experimental protocol for WOODPUSH was similar to that of CONNECT FOUR. The first list of problems corresponds to positions occuring after 4 moves on a board of length 8 with 3 pieces for each player. The second list of problems corresponds to positions occuring after 8 moves on a board of length 13 with 2 pieces for each player. Table 3.2 presents aggregates data for the solving time and the number of node creations, while Figure 3.5 presents the number of node creations for each problem in the second list.

In WOODPUSH $(8, 3)$, it is possible to create final positions with scores ranging from -18 to 18 but these positions might not be accessible from the start position. Indeed, in our experiments, no final position with a score below -5 or over 5 was ever reached. However, while the scores remained between -5 and 5, the exact range varied depending on the problem. While doing a binary search for the outcome is the natural generic process for solving a multi-outcome game with PNS, we decided to compare MOPNS to the ideal case for PNS which only involves two runs per position. On the other hand, we only assumed for MOPNS that the outcome was in $[-5, 5]$. Therefore, the results presented in Table 3.2 and Figure 3.5 significantly favour PNS.

Tables 3.3 and 3.4 detail the results for the position presented in Figure 3.6.

Table 3.2: Cumulated time and number of node creation for the MOPNS and PNS algorithms in the game of WOODPUSH.

		MOPNS	PNS
Size $(8, 3)$, 99 positions after 4 moves	Total time (seconds)	718	702
	Total node creations	31,328,178	34,869,213
	Lowest time	25	74
	Lowest node creations	76	23
Size $(13, 2)$, 256 positions after 8 moves	Total time (seconds)	4796	4573
	Total node creations	155,756,022	174,285,199
	Lowest time	98	158
	Lowest node creations	205	51

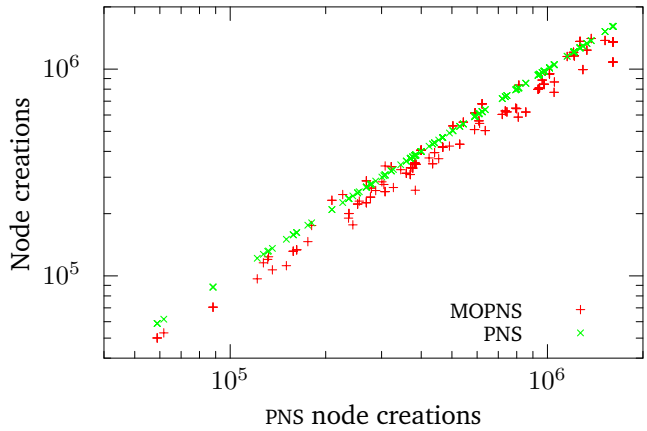


Figure 3.5: Comparison of the number of node creations for MOPNS and PNS for solving 256 WOODPUSH positions on size $(13, 2)$.

Table 3.3: Detailed results for PNS on the 86th WOODPUSH problem of size $(8, 3)$.

	PNS								
Setting	≥ -4	≥ -3	≥ -2	≥ -1	≥ 1	≥ 2	≥ 3	≥ 4	≥ 5
Time	0.508	0.500	0.884	1.188	1.200	1.204	3.084	1.360	1.356
Nodes	39340	39340	68035	84184	84568	84545	178841	98069	98069
Result	true	true	true	true	true	true	false	false	false

Table 3.4: Detailed results for the multi-outcome algorithms on the 86th WOODPUSH problem of size $(8, 3)$.

	Dichotomic PNS		MOPNS	
Setting	$[-5, 5]$	$[1, 3]$	$[-5, 5]$	$[1, 3]$
Time	6.676	4.288	4.556	3.684
Nodes	351366	263386	210183	191127
Result	2	2	2	2

Figure 3.6: 86th WOODPUSH problem on size $(8, 3)$.

The PNS tree did not access any position with a score lower or equal to -4 nor any position with a score greater or equal to 5.

3.9 Conclusion and discussion

We have presented a generalized Proof Number algorithm that solves games with multiple outcomes in one run. Running PNS multiple times to prove an outcome develops the same nodes multiple times while in MOPNS these nodes are developed only once. MOPNS has been formally proved equivalent to PNS in two-outcome games and we have shown how safe pruning could be performed in multiple outcome games. For small CONNECT FOUR and WOODPUSH boards, in most cases MOPNS solves the games with fewer node creations than PNS even if it already knows the optimal outcome of the game and no binary search is needed.

Conspiracy numbers search [97, 133] also deals with a range of possible

evaluations at the leaves of the search tree. However, the algorithm works with a heuristic evaluation function whereas MOPNS has no evaluation function and only scores solved positions. Moreover the development of the tree is not the same for MOPNS and for Conspiracy numbers search since MOPNS tries to prove the outcome that costs the less effort whereas Conspiracy numbers search tries to eliminate unlikely values of the evaluation function.

The Iterative PNS algorithm [98] also deals with multiple outcomes but uses the usual proof and disproof numbers as well as a value for each node and a cache. The main difference between Iterative PNS and the proposed MOPNS, is that Iterative PNS tries to find the value of the game by eliminating outcomes step by step. On the other hand, MOPNS can dynamically focus on newly promising values even if previously promising values have not been completly outruled yet.

We have assumed in this thesis that the game structure was unfolded into a tree. In most practical cases it actually is a DAG and in some cases the graph contains cycles.[2] The theoretical results presented in this article still hold in the DAG case, provided the definition of the relevancy bounds is adapted to reflect the fact that a node may have multiple parents and some of them might not yet be in the tree. The double count problem of PNS will also affect MOPNS in DAGs, but it is possible to take advantage of previous work on the handling of transpositions in PNS [139, 100]. Similarly, the problems encountered by MOPNS in cyclic graphs are similar to that of PNS and DFPN in cyclic graphs. Fortunately, it should be straightforward to adapt Kishimoto and Müller's ideas [73] from DFPN to a depth-first version of MOPNS.

In future work, we plan on trying to adapt the PN^2 paralellization scheme suggested by Saffidine et al. [126] to games with multiple outcomes via MOPNS. We would also like to study a depth-first version of MOPNS that can be obtained via Nagai's transformation [103].

Finally, studying how MOPNS can be extended to deal with problems where the outcome space is not known beforehand or is continuous in order to develop an effort number algorithm for non-deterministic two-player games is definitely an attractive research agenda.

[2] For instance, the original rules for CHESS result in a DAG because of the 50-moves rule, but this rule is usually abstracted away, resulting in a cyclic structure.

4 Modal Logic K Model Checking

In this chapter, we investigate the relationship between Multi-agent Modal Logic K (MMLK) and sequential game search. Drawing inspiration from game search algorithms such as MCTS, PNS, or A, we suggest several new model checking algorithms for MMLK. We prove that one of these algorithms, Minimal Proof Search (MPS), allows to find minimal witness/counterexample for the model checking problem optimally.*

We show how to express formally multiple solution concepts of sequential games in MMLK. Indeed, the testing of many solution concepts on sequential games can be seen a model cheking problem for MMLK in disguise. Finally, we use the MMLK model checking framework to obtain a classification of more than a dozen game tree search algorithms.

This Chapter includes results from the following papers.

[122] Abdallah Saffidine and Tristan Cazenave. A general multi-agent modal logic K framework for game tree search. In Computer Games Workshop @ ECAI*, Montpellier, France, August 2012*

[119] Abdallah Saffidine. Minimal proof search for modal logic K model checking. In Luis del Cerro, Andreas Herzig, and Jérôme Mengin, editors, 13th European Conference on Logics in Artificial Intelligence (JELIA)*, volume 7519 of* Lecture Notes in Computer Science*, pages 346–358. Springer, Berlin / Heidelberg, September 2012. ISBN 978-3-642-33352-1*

Contents

4.1 Introduction

Model checking for temporal logics such as LTL or CTL is a major research area with important applications in software and hardware verification [32]. Model checking for agent logics such as ATL or S5 is now also regarded as an important topic with a variety of applications [158, 161, 90]. On the other hand, Modal Logic K is usually considered the basis upon which more elaborate modal logics are built, such as S5, PDL, LTL, CTL, or ATL [14, 143]. Multi-agent Modal Logic

K (MMLK) can also be used directly to model (sequential) perfect information games.

A natural question in perfect information games is indeed whether some agent can achieve a specified goal from a given position. The other agents can either be assumed to be cooperative, or adversarial. For example, an instance of such a question in CHESS is: "Can *White* force a capture of the black Queen in exactly 5 moves?" In CHINESE CHECKERS, we could ask whether one player can force a win within ten moves. Ladder detection in GO and *helpmate* solving in CHESS also belong to this framework. The latter is an example of a cooperative situation.

While And/Or trees are as expressive as the combination of MMLK and Game Automata (GAs), we believe that the separation of concerns between the logic and the Game Automaton is beneficial in practice. For instance, if the properties to be checked are encoded in the logic rather than in the graph, there is no need to rewrite the rules of CHESS if one is interested in finding helpmates instead of checkmates, or if one just wants to know if any piece can be captured in two moves from a given position. The encoding through an And/Or graph would be different in every such situation while in our approach, only the modal logic formula needs to be adapted.

The first contribution in this chapter is a formal definition of (dis)proof in MMLK model checking, as well as a very general definition of (dis)proof cost.[1]

We then provide with a variety of new algorithms to solve MMLK model checking problem (Section 4.3). These algorithms are based on the depth-first search and the best-first search approaches that we have seen in Chapter 2 and that we adapt to the setting of this chapter. They include a generalization of Proof Number Search (PNS), the practical importance of which has been stressed already, and an algorithm inspired by Monte Carlo Tree Search (MCTS). To do so, we extend the concept of Monte Carlo playouts which are generalized into Monte Carlo probes (Section 4.3.4). Finally, we develop Minimal Proof Search (MPS), a model checking algorithm that outputs (dis)proofs of minimal size for the proposed broad definition of (dis)proof cost. Besides proving the correctness and admissibility of MPS, we also argue that it is optimal.

[1] Following the convention in Proof Number Search, we use the term proof and disproof instead of witness and counterexample which are more common in the model checking literature.

In Section 4.5, we show that many abstract properties of games can be formally expressed as MMLK formulas. We tighten the correspondence between sequential games and MMLK model checking in Section 4.6 by showing that numerous previous game tree search algorithms can be directly expressed as combinations of model checking problems and model checking algorithms (Section 4.6).

We demonstrate that the MMLK allows new solution concepts to be rigorously defined and conveniently expressed. Moreover, many new algorithms can be derived through new combinations of the proposed search algorithms and existing or new solution concepts (formulas). Finally, it is a convenient formal model to prove properties about game algorithms.

We believe that these contributions can be of interest to a broad class of researchers. Indeed, the model checking algorithms we develop for MMLK could serve as a basis for model checking algorithms for more elaborate logics such as LTL, CTL, and ATL. The games that fall under our formalism constitute a significant fragment of the games encountered in General Game Playing (GGP) [55]. We also express a generalization of the MCTS algorithm that can be used even when not looking for a winning strategy. Finally, the unifying framework we provide makes understanding a wide class of game tree search algorithms relatively easy, and the implementation is straightforward.

4.2 Definitions

We define in this section various formal objects that will be used throughout the chapter. The GA is the underlying system which is to be formally verified. The MMLK is the language to express the various properties we want to model check GAs against. Finally, a (dis)proof is a tree structure that shows whether a property is true on a state in a GA.

4.2.1 Game model

We now define the model we use to represent games. We focus on a subset of the strategy games that are studied in Game Theory. The games we are interested in are turn-based games with perfect and complete information. Despite these restrictions, the class of games considered is quite large, including classics such

as CHESS and GO, but also multiplayer games such as CHINESE CHECKERS, or single player games such as SOKOBAN.

A GA is a kind of labelled transition system where both the states and the transitions are labelled. If a GA is interpreted as a perfect information game, then the states of the game automaton correspond to possible positions over the board, a transition corresponds to a move from one position to the next and its label is the player making that move. The state labels are domain specific information about states, for instance we could have a label for each triple (piece, owner, position) in CHESS-like games. The formal definition of GAs is almost exactly that of transition systems (see Definition 1).

Definition 21. A *Game Automaton* is a 5-tuple $G = \langle L, R, S, \lambda, \delta \rangle$ with the following components:

- L is a non-empty set of *atoms* (or state labels);
- R is a non-empty finite set of *agents* (or transition labels, or *players*);
- S is a set of *game states*;
- $\lambda : S \to 2^L$ maps each state q to its labels;
- $\delta : S \times R \to 2^S$ is a transition function that maps a state and an agent to a set of next states.

In the following, we will use p, p', p_1, ... for atoms, a for an arbitrary agent, and q, q', q_1, ... for game states. We write $q \xrightarrow{a} q'$ when $q' \in \delta(q, a)$ and we read *agent a can move from q to q'*. We understand δ as: in a state q, agent a is free to choose as the next state any q' such that $q \xrightarrow{a} q'$. Note that δ returns the set of successors, so it need not be a partial function to allow for states without successors. If an agent a has no moves in a state q, we have $\delta(q, a) = \emptyset$.

Note that we do not require the GA to define δ such that $a \neq a'$ implies $\delta(q, a) = \emptyset$ or $\delta(q, a') = \emptyset$. Although the games are sequential, we do not assume that positions are tied to a player on turn. This is natural for some games such as GO or HEX. If the turn player is tightly linked to the position, we can simply consider that the other players have no legal moves, or we can add a *pass* move for the other players that will not change the position.

We do not mark final states explicitly, neither do we embed the concept of game outcome and reward explicitly in the previous definition. We rather rely on a labelling of the states through atomic propositions. For instance, we can imagine having an atomic proposition for each possible game outcome and label each final state with exactly one such proposition.

4.2.2 Multi-agent Modal Logic K

Modal logic is often used to reason about the knowledge of agents in a multi-agent environment [14]. In such environments, the states in the GA are interpreted as possible worlds and additional constraints are put on the transition relation which is interpreted through the concepts of knowledge or belief. In this work, though, the transition relation is interpreted as a *legal move* function, and we do not need to put additional constraints on it. Since we do not want to reason about the epistemic capacities of our players, we use the simplest fragment of multi-agent modal logic [14].

Following loosely [14], we define the Multi-agent Modal Logic K over a set of atoms L as the formulas we obtain by combining the negation and conjunction operators with a set of *box* operators, one per agent.

Definition 22. The set of well-formed *Multi-agent Modal Logic K (MMLK)* formulas over L and R is defined through the following grammar.

$$\phi := \quad p \quad | \quad \neg\phi \quad | \quad \phi \wedge \phi \quad | \quad \Box_a \, \phi$$

Thus, a formula is either an atomic proposition, the negation of a formula, the conjunction of two formulas, or the modal operator $\Box_a$ for a player a applied to a formula. In the following, ϕ, ϕ', ϕ_1,... stand for arbitrary MMLK formulas. We define the usual syntactic shortcuts for the disjunction $\phi_1 \vee \phi_2 \stackrel{\text{def}}{=} \neg(\neg\phi_1 \wedge \neg\phi_2)$, and for the existential modal operators $\Diamond_a \, \phi \stackrel{\text{def}}{=} \neg \, \Box_a \, \neg\phi$. The precedence of $\Diamond_a$ and $\Box_a$, for any agent a, is higher than $\vee$ and $\wedge$, that is, $\Diamond_a \, \phi_1 \vee \phi_2 = (\Diamond_a \, \phi_1) \vee \phi_2$.

The box operators convey necessity and the diamond operators convey possibility: $\Box_a \, \phi$ can be read as *it is necessary for agent a that ϕ*, while $\Diamond_a \, \phi$ is *it is possible for a that ϕ*.

4.2.3 The Model Checking Problem

We can now interpret MMLK formulas over GAs via the satisfaction relation $\models$. Intuitively, a state in a GA constitutes the context of a formula, while a formula constitutes a property of a state. A formula might be satisfied in some contexts and not satisfied in other contexts, and some properties hold in a state while others do not. Determining whether a given formula ϕ holds in a given state q (in a given implicit GA) is what is commonly referred to as *the model checking problem*. If it is the case, we write $q \models \phi$, otherwise we write $q \not\models \phi$.

It is possible to decide whether $q \models \phi$ by examining the structure of ϕ, the labels of q, as well as the accessible states.

Definition 23. The formulas *satisfied* by a state q can be constructed by induction as follows.

- If p is a label of q, that is if $p \in \lambda(q)$, then $q \models p$;
- if $q \not\models \phi$ then $q \models \neg\phi$;
- if $q \models \phi_1$ and $q \models \phi_2$ then $q \models \phi_1 \wedge \phi_2$;
- if for all q' such that $q \xrightarrow{a} q'$, we have $q' \models \phi$, then $q \models \Box_a \phi$.

It can be shown that the semantics for the syntactic shortcuts defined previously behave as expected. $q \models \phi_1 \vee \phi_2$ if and only if $q \models \phi_1$ or $q \models \phi_2$; $q \models \Diamond_a \phi$ if there exists a q' such that $q \xrightarrow{a} q'$ and $q' \models \phi$.

This semantical interpretation of MMLK allow an alternative understanding of the box and diamond operators. We can also read $\Box_a \phi$ as *all moves for agent a lead to states where ϕ holds* and read $\Diamond_a \phi$ as *there exists a move for agent a leading to a state where ϕ holds*.

4.2.4 Proofs and Counterexamples

In practice, we never explicitly construct the complete set of formulas satisfied by a state. So when some computation tells us that a formula ϕ is indeed (not) satisfied by a state q, some sort of evidence might be desirable. In software model checking, a model of the program replaces the GA, and a formula in a temporal logic acts as a specification of the program. If a correct model checker asserts that the program does not satisfy the specification, it means that the

program or the specification contained a bug. In those cases, it can be very useful for the programmers to have access to an evidence by the model checker of the mismatch between the formula and the system as it is likely to lead them to the bug.

In this section we give a formal definition of what constitutes a *proof* or a *disproof* for the class of model checking problems we are interested in. It is possible to relate the following definitions to the more general concept of *tree-like* counterexamples used in model checking ACTL [33].

Definition 24. An *exploration tree* for a formula ϕ in a state q is a tree with root n associated with a pair (q, ϕ) with q a state and ϕ a formula, such that n satisfies the following properties.

- If n is associated with (q, p) with $p \in L$, then it has no children;
- if n is associated with $(q, \neg\phi)$ then n has at most one child and it is an exploration tree associated with (q, ϕ);
- if a node n is associated with $(q, \phi_1 \wedge \phi_2)$ then any child of n (if any) is an exploration tree associated with (q, ϕ_1) or with (q, ϕ_2);
- if a node n is associated with $(q, \Box_a \phi)$ then any child of n (if any) is an exploration tree associated with (q', ϕ) for some q' such that $q \xrightarrow{a} q'$.
- In any case, no two children of n are associated with the same pair.

Unless stated otherwise, we will not distinguish between a tree and its root node. In the rest of the paper, n, n', n_1, ... will be used to denote nodes in exploration trees.

Definition 25. A *proof* (resp. a *disproof*) that $q \models \phi$ is an exploration tree with a root n associated with (q, ϕ) satisfying the following hypotheses.

- If $\phi = p$ with $p \in L$, then $p \in \lambda(q)$ (resp. $p \notin \lambda(q)$);
- if $\phi = \neg\phi'$, then n has exactly one child n' and this child is a disproof (resp. proof);
- if $\phi = \phi_1 \wedge \phi_2$, then n has exactly two children n_1 and n_2 such that n_1 is a proof that $q \models \phi_1$ and n_2 is a proof that $q \models \phi_2$ (resp. n has exactly one child n' and n' is a disproof that $q \models \phi_1$ or n' is a disproof that $q \models \phi_2$);

- if $\phi = \Box_a \phi'$, then n has exactly one child n' for each $q \xrightarrow{a} q'$, and n' is a proof for $q' \models \phi'$ (resp. n has exactly one child n' and n' is a disproof for $q' \models \phi'$ for some $q \xrightarrow{a} q'$).

4.2.5 Cost Functions

To remain as general as possible with respect to the definitions of a *small* (dis)proof in the introduction, we introduce a cost function k as well as cost aggregators $A_\wedge$ and $A_\Box$. These functions can then be instantiated in a domain dependent manner to get the optimal algorithm for the domain definition of minimality. This approach has been used before in the context of A* and AO* [111].

We assume given a *base cost function* $k : L \to \mathbb{R}^+$, as well as a *conjunction cost aggregator* $A_\wedge : \mathbb{N}^{\mathbb{R}^+ \cup \{\infty\}} \to \mathbb{R}^+ \cup \{\infty\}$ and a *box modal cost aggregator* $A_\Box : \Sigma \times \mathbb{N}^{\mathbb{R}^+ \cup \{\infty\}} \to \mathbb{R}^+ \cup \{\infty\}$, where $\mathbb{N}^{\mathbb{R}^+ \cup \{\infty\}}$ denotes the set of multisets of $\mathbb{R}^+ \cup \{\infty\}$.

We assume the aggregators are increasing in the sense that adding elements to the input increases the cost. For all costs $x \leq y \in \mathbb{R}^+ \cup \{\infty\}$, multisets of costs $X \in \mathbb{N}^{\mathbb{R}^+ \cup \{\infty\}}$, and for all agents a, we have for the conjunction cost aggregator $A_\wedge(X) \leq A_\wedge(\{x\} \cup X) \leq A_\wedge(\{y\} \cup X)$, and for the box aggregator $A_\Box(a, X) \leq A_\Box(a, \{x\} \cup X) \leq A_\Box(a, \{y\} \cup X)$.

We further assume that aggregating infinite costs results in infinite costs and that aggregating finite numbers of finite costs results in finite costs. For all costs $x \in \mathbb{R}^+$, multisets of costs $X \in \mathbb{N}^{\mathbb{R}^+ \cup \{\infty\}}$, and for all agents a, $A_\wedge(\{\infty\}) = A_\Box(a, \{\infty\}) = \infty$ and that $A_\wedge(X) < \infty \Rightarrow A_\wedge(\{x\} \cup X) < \infty$ and $A_\Box(a, X) < \infty \Rightarrow A_\Box(a, \{x\} \cup X) < \infty$.

Note that in our presentation, there is no cost to a negation. The justification is that we want a proof aggregating over a disjunction to cost as much as a disproof aggregating over a conjunction with children of the same cost, without having to include the disjunction and the diamond operator in the base syntax.

Given k, $A_\wedge$, and $A_\Box$, it is possible to define the *global cost function* for a (dis)proof as shown in Table 4.1.

Example 8. Suppose we are interested in the nested depth of the $\Box$ operators in the (dis)proof. Then we define $k = 0$, $A_\wedge = \max$, and $A_\Box(a, X) = 1 + \max X$ for all a.

Table 4.1: Cost K of a proof or a disproof for a node n as a function of the base cost function k and the aggregators $A_\wedge$ and $A_\Box$. C is the set of children of n.

Label of n	Children of n	$K(n)$
(q, p)	$\emptyset$	$k(p)$
$(q, \neg\phi)$	$\{c\}$	$K(c)$
$(q, \phi_1 \wedge \phi_2)$	C	$A_\wedge(\{K(c) \mid c \in C\})$
$(q, \Box_a \phi)$	C	$A_\Box(a, \{K(c) \mid c \in C\})$

Example 9. Suppose we are interested in the number of atomic queries to the underlying system (the GA). Then we define $k = 1$, $A_\wedge(X) = \sum X$, and $A_\Box(a, X) = \sum X$ for all a.

Example 10. Suppose we are interested in minimizing the amount of expansive interactions with the underlying system. Then we define $A_\wedge(X) = \sum X$, and $A_\Box(a, X) = k_{\Box_a} + \sum X$ for all a. In this case, we understand that $k(p)$ is the price for querying p in any state, and $k_{\Box_a}$ is the price for getting access to the transition function for agent a in any state.

4.3 Model Checking Algorithms

We now define several model checking algorithms. That is, we present algorithms that allow to decide whether a state q satisfies a formula ϕ ($q \models \phi$).

4.3.1 Depth First Proof Search

Checking whether a formula is satisfied on a state can be decided by a depth-first search on the game tree as dictated by the semantics given in Section 4.2.2. Pseudo-code for the resulting algorithm, called Depth First Proof Search (DFPS) is presented in Algorithm 7.

4.3.2 Best-first Search Algorithms

We can propose several alternatives to the DFPS algorithm to check a given formula in a given state. We adapt the generic Best First Search (BFS) framework proposed in Chapter 2 to express model checking algorithms. Best-first search

Algorithm 7: Pseudo-code for the DFPS algorithm.

```
dfps(state q, formula φ)
  switch on the shape of φ do
    case p ∈ L  return p ∈ λ(q)
    case φ1 ∧ φ2  return dfps(q, φ1) ∧ dfps(q, φ2)
    case ¬φ1  return ¬ dfps(q, φ1)
    case □a φ1
      foreach q′ in {q′, q -a→ q′} do
        if not dfps(q′, φ1) then return false
      return true
```

algorithms must maintain a partial tree in memory, the shape of which is determined by the formula to be checked.

Nodes are mapped to a (state q, formula ϕ) label. A leaf is terminal if its label is an atomic proposition $p \in \lambda$ otherwise it is non-terminal. Each node is associated to a unique position, but a position may be associated to multiple nodes. [2]

The following static observations can be made about partial trees:

- an internal node labelled $(q, \neg\phi)$ has exactly one child and it is labelled (q, ϕ);
- an internal node labelled $(q, \phi_1 \wedge \phi_2)$ has exactly two children which are labelled (q, ϕ_1) and (q, ϕ_2);
- an internal node labelled $(q, \Box_a \phi)$ has as many children as there are legal transition for a in q. Each child is labelled (q', ϕ) where q' is the corresponding state.

The generic framework is described in Algorithm 8. An instance must provide a data type for node specific information which we call *node value* and the following procedures. The `info-term` defines the value of terminal leaves. The `init-leaf` procedure is called when initialising a new leaf. The `update`

[2] While it is possible to store the state q associated to a node n in memory, it usually is more efficient to store move information on edges and reconstruct q from the root position and the path to n.

Algorithm 8: Pseudo-code for a best-first search algorithm.

```
extend(node n)
  switch on the shape of n.formula do
    case φ1 ∧ φ2
      foreach i in {1, 2} do
        n_i ← new node
        n_i.state ← q ; n_i.formula ← φ_i ; n_i.info ← ζ(q, φ_i)
        Add n_i as child_i of n
    case ¬φ
      n' ← new node
      n'.state ← q ; n'.formula ← φ ; n'.info ← ζ(q, φ)
      Add n' as the child of n
    case □_a φ
      foreach q' in {q', n.state -a-> q'} do
        n' ← new node
        n'.state ← q' ; n'.formula ← φ ; n'.info ← ζ(q', φ)
        Add n' to n.children

backpropagate(node n)
  old_info ← n.info
  switch on the shape of n.formula do
    case φ1 ∧ φ2   n.info ← H_∧(n.child_1, n.child_2)
    case ¬φ        n.info ← H_¬(n.child)
    case □_a φ     n.info ← H_□(n.children)
  if old_info = n.info ∨ n = r then return n
  else return backpropagate(n.parent)

bfs(state q, formula φ)
  r ← new node
  r.state ← q ; r.formula ← φ ; r.info ← ζ(q, φ)
  n ← r
  while r.info ∉ S do
    while n is not a leaf do
      switch on the shape of n.formula do
        case φ1 ∧ φ2   n ← max_{≼_∧^{n.info}} {n.child_1, n.child_2}
        case ¬φ        n ← n.child
        case □_a φ     n ← max_{≼_□^{n.info}} {n.children}
      n ← select-child(n)
    extend(n)
    n ← backpropagate(n)
  return r
```

procedure determines how the value of an internal node evolves as a function of its label and the value of the children. The `select-child` procedure decides which child is best to be explored next depending on the node's value and label and the value of each child. We present possible instances in Sections 4.3.3 and 4.3.4.

The `backpropagate` procedure implements a small optimization known as the *current node enhancement* [4]. Basically, if the information about a node n are not changed, then the information about the ancestors of n will not change either and so the next descend will reach n. Thus, it is possible to shortcut the process and start the next descent at n directly.

4.3.3 Proof Number Proof Search

We present a first instance of the generic best-first search algorithm described in Section 4.3.2 under the name Proof Number Proof Search (PNPS). This algorithm uses the concept of *effort numbers* and is inspired from Proof Number Search (PNS) [4, 155].

The node specific information needed for PNPS is a pair of numbers which can be positive, equal to zero, or infinite. We call them *proof number* (p) and *disproof number* (d). Basically, if a subformula ϕ is to be proved in a state s and n is the corresponding node in the constructed partial tree, then the p (resp. d) in a node n is a lower bound on the number of nodes to be added to the tree to be able to exhibit a proof that $s \models \phi$ (resp. $s \not\models \phi$). When the p reached 0 (and the d reaches ∞), the fact has been proved and when the p reached ∞ (and the d reaches 0) the fact has been disproved.

The `info-term` and `init-leaf` procedures are described in Table 4.2, while Table 4.3 and 4.4 describe the `update` and `select-child` procedures, respectively. If domain specific information is available, we can initialize the p and d in `init-leaf` with heuristical values.

4.3.4 Monte Carlo Proof Search

MCTS is a recent game tree search technique based on multi-armed bandit problems [20]. MCTS has enabled a huge leap forward in the playing level of artificial GO players. It has been extended to prove wins and losses under

Table 4.2: Initial values for leaf nodes in PNPS.

	Node label	p	d
`info-term`	(q, p) when $p \in \lambda(q)$	0	∞
	(q, p) when $p \notin \lambda(q)$	∞	0
`init-leaf`	(q, ϕ)	1	1

Table 4.3: Determination of values for internal nodes in PNPS.

Node label	Children	p	d
$(q, \neg\phi)$	$\{c\}$	$d(c)$	$p(c)$
$(q, \phi_1 \wedge \phi_2)$	C	$\sum_C p$	$\min_C d$
$(q, \Box_a \phi)$	C	$\sum_C p$	$\min_C d$

Table 4.4: Selection policy for PNPS.

Node label	Children	Chosen child
$(q, \neg\phi)$	$\{c\}$	c
$(q, \phi_1 \wedge \phi_2)$	C	$\arg\min_C d$
$(q, \Box_a \phi)$	C	$\arg\min_C d$

the name MCTS Solver [165] and it can be seen as the origin of the algorithm presented in this section which we call Monte Carlo Proof Search (MCPS).

The basic idea in MCPS is to evaluate whether a state s satisfies a formula via probes in the tree below s. Monte Carlo probes are a generalization of Monte Carlo playouts used in MCTS. A Monte Carlo playout is a random path of the tree below s, whereas a Monte Carlo probe is a random subtree with a shape determined by an MMLK formula. A probe is said to be *successful* if the formulas at the leaves are satisfied in the corresponding states. Determining whether a new probe generated on the fly is successful can be done as demonstrated in Algorithm 9.

Like MCTS, MCPS explores the GA in a best first way by using aggregates of information given by the playouts. For each node n, we need to know the total number of probes rooted below n (denoted by t) and the number of successful probes among them (denoted by r). We are then faced with an exploration-

Algorithm 9: Pseudo-code for a Monte Carlo Probe.

```
probe(state q, formula φ)
  switch on the shape of φ do
    case p ∈ L
      return p ∈ λ(q)
    case φ1 ∧ φ2
      return probe(q, φ1) ∧ probe(q, φ2)
    case ¬φ1
      return ¬ probe(q, φ1)
    case □a φ1
      q' ← random state such that q -a-> q'
      return probe(q', φ1)
```

Table 4.5: Initialisation for leaf values in MCPS for a node n.

	Node label	s	r	t
`info-term`	(q, p) where $p \in \lambda(q)$	$\top$	1	1
	(q, p) where $p \notin \lambda(n)$	$\bot$	0	1
`init-leaf`	(q, ϕ)	?	`probe`(q, ϕ)	1

exploitation dilemma between running probes in nodes which have not been explored much (t is small) and running probes in nodes which seem successful (high $\frac{r}{t}$ ratio). This concern is addressed using the UCB formula [20].

Similarly to MCTS Solver, we will add another label to the value of nodes called s. s represents the proof status and allows to avoid solved subtrees. s can take three values: $\top$, $\bot$, or ?. These values respectively mean that the corresponding subformula was proved, disproved, or neither proved nor disproved for this node.

We describe the `info-term`, `init-leaf`, `update`, and `select-child` procedures in Table 4.5, Table 4.6, and Table 4.7.

4.4 Minimal Proof Search

Let $q \models \phi$ be a model checking problem and n_1 and n_2 two proofs as defined in Section 4.2.4. Even if n_1 is not a subtree of n_2, there might be reasons to prefer,

Table 4.6: Determination of values for internal nodes in MCPS.

Node label	Children	s	r	t
$(q, \neg\phi)$	$\{c\}$	$\neg s(c)$	$t(c) - r(c)$	$t(c)$
$(q, \phi_1 \wedge \phi_2)$	C	$\bigwedge_C s$	$\sum_C r$	$\sum_C t$
$(q, \Box_a \phi)$	C	$\bigwedge_C s$	$\sum_C r$	$\sum_C t$

Table 4.7: Selection policy for MCPS in a node n.

Node label	Children	Chosen child
$(q, \neg\phi)$	$\{c\}$	c
$(q, \phi_1 \wedge \phi_2)$	C	$\arg\max_{C, s(c)=?} \frac{t-r}{t} + \sqrt{\frac{2 \ln t(n)}{t}}$
$(q, \Box_a \phi)$	C	$\arg\max_{C, s(c)=?} \frac{t-r}{t} + \sqrt{\frac{2 \ln t(n)}{t}}$

n_1 over n_2. For instance, we can imagine that n_1 contains fewer nodes than n_2, or that the depth of n_1 is smaller than that of n_2.

In this chapter, we put forward a model checking algorithm for MMLK that we call Minimal Proof Search (MPS). As the name indicates, given a model checking problem $q \models \phi$, the MPS algorithm outputs a proof that q satisfies ϕ or a counterexample, this proof/counterexample being minimal for some definition of size. Perfect information games provide at least two motivations for small proofs. In game playing, people are usually interested in "short" proofs, for instance a CHESS player would rather deliver checkmate in three moves than in nine moves even if both options grant them the victory. In game solving, "compact" proofs can be stored and independently checked efficiently.

Our goal is related both to heuristic search and software model checking. On one hand, the celebrated A* algorithm outputs a path of minimal cost from a starting state to a goal state. This path can be seen as the proof that the goal state is reachable, and the cost of the path is the size of the proof. On the other hand, finding small counterexamples is an important subject in software model checking. For a failure to meet a specification often indicates a bug in the program, and a small counterexample makes finding and correcting the bug easier [59].

Like A*, MPS is optimal, in the sense that any algorithm provided with the

Table 4.8: Definition of the heuristic functions I and J.

Shape of ϕ	$I(\phi)$	$J(\phi)$
p	$k(p)$	$k(p)$
$\neg\phi'$	$J(\phi')$	$I(\phi')$
$\phi_1 \wedge \phi_2$	$A_\wedge(\{I(\phi_1), I(\phi_2)\})$	$\min_{i\in\{1,2\}} A_\wedge(\{J(\phi_i)\})$
$\Box_a\, \phi'$	$A_\Box(a, \emptyset)$	$A_\Box(a, \{J(\phi')\})$

same information and guaranteed to find a proof of minimal size needs to do as many node expansions as MPS.

4.4.1 Heuristics

We define two *heuristic* functions I and J to estimate the minimal amount of interaction needed with the underlying system to say anything about a formula ϕ. These functions are defined in Table 4.8, $I(\phi)$ is a lower bound on the minimal amount of interaction to prove ϕ and $J(\phi)$ is a lower bound on the minimal amount of interaction to disprove ϕ.

The heuristics I and J are *admissible*, that is, they never overestimate the cost of a (dis)proof.

Proposition 23. *Given a formula ϕ, for any state q, for any proof n that $q \models \phi$ (resp. disproof), $I(\phi) \leq K(n)$ (resp. $J(\phi) \leq K(n)$).*

Proof. We proceed by structural induction on the shape of formulas. For the base case $\phi = p$, if n is a proof that $q \models p$, then the n label of n is (q, p) and its cost is $K(n) = k(p)$, which is indeed greater or equal to $I(p) = J(p) = k(p)$.

For the induction case, take the formulas ϕ_1 and ϕ_2 and assume that for any proofs (resp. disproofs) n_1 and n_2, the cost is greater than the heuristic value: $I(\phi_1) \leq K(n_1)$ and $I(\phi_2) \leq K(n_2)$ (resp. $J(\phi_1) \leq K(n_1)$ and $J(\phi_2) \leq K(n_2)$).

For any proof (resp. disproof) n with label $(q, \neg\phi_a)$ and child c, the cost of n is the cost of the disproof (resp. proof) c: $K(n) = K(c)$. The disproof (resp. proof) c is associated with (q, ϕ_1) and we know from the induction hypothesis that $J(\phi_1) \leq K(c)$ (resp. $I(\phi_1) \leq K(c)$). By definition of the heuristics, $I(\phi) = J(\phi_1)$ (resp. $J(\phi) = I(\phi_1)$), therefore we have $I(\phi) \leq K(n)$ (resp. $J(\phi) \leq K(n)$).

For any proof (resp. disproof) n with label $(q, \phi_1 \wedge \phi_2)$ and children c_1, c_2 (resp. child c), the cost of n is the sum of the costs of the children: $K(n) = K(c_1) + K(c_2)$ (resp. $K(n) = K(c)$). The nodes c_1 and c_2 are associated with (q, ϕ_1) and (q, ϕ_2) (resp. c is associated with (q, ϕ_1) or to (q, ϕ_2)) and we know from the induction hypothesis that $I(\phi_1) \leq K(c_1)$ and $I(\phi_2) \leq K(c_2)$ (resp. $J(\phi_1) \leq K(c)$ or $J(\phi_2) \leq K(c)$). By definition of the heuristics, $I(\phi) = I(\phi_1)+I(\phi_2)$ (resp. $J(\phi) = \min\{J(\phi_1), J(\phi_2)\}$), therefore we have $I(\phi) \leq K(n)$ (resp. $J(\phi) \leq K(n)$).

The remaining case is very similar and is omitted. □

Lemma 1. *For any formula ϕ, $I(\phi) < \infty$ and $J(\phi) < \infty$.*

Proof. We proceed by structural induction on ϕ. For the base case, $\phi = p$, simply recall that the range of k is $\mathbb{R}^+$. The induction case results directly from the assumptions on the aggregators. □

We inscribe the MPS algorithm in a best first search framework inspired by game tree search. We then specify a function for initializing the leaves, a function to update tree after a leaf has been expanded, a selection function to decide which part of the tree to expand next, and a stopping condition for the overall algorithm.

Algorithm 8 develops an exploration tree for a given state q and formula ϕ. To be able to orient the search efficiently towards proving or disproving the model checking problem $q \models \phi$ instead of just exploring, we need to attach additional information to the nodes beyond their (state, formula) label. This information takes the form of two *effort* numbers, called the *minimal proof number* and *minimal disproof number*. Given a node n associated with a pair (q, ϕ), the minimal proof number of n, $\mathrm{MPN}(n)$, is an indication on the cost of a proof for $q \models \phi$. Conversely, the minimal disproof number of n, $\mathrm{MDN}(n)$, is an indication on the cost of a disproof for $q \models \phi$. For a more precise relationship between $\mathrm{MPN}(n)$ and the cost of a proof see Prop. 28.

The algorithm stops when the minimal (dis)proof number reaches ∞ as it corresponds to the exploration tree containing a (dis)proof of minimal cost (see Prop. 26).

The values for the effort numbers in terminal leaves and in newly created leaves are defined in Table 4.9. The values for the effort numbers of an internal

Table 4.9: Values for terminal nodes and initial values for leaves.

	Node label	MPN	MDN
`info-term`	(q,p) where $p \in \lambda(q)$	$k(p)$	∞
	(q,p) where $p \notin \lambda(q)$	∞	$k(p)$
`init-leaf`	(q,ϕ)	$I(\phi)$	$J(\phi)$

Table 4.10: Determination of values for internal nodes.

Node label	Children	MPN	MDN
$(q,\neg\phi)$	$\{c\}$	$\mathrm{MDN}(c)$	$\mathrm{MPN}(c)$
$(q,\phi_1 \wedge \phi_2)$	C	$A_\wedge(\{\mathrm{MPN}(c) \mid c \in C\})$	$\min_C A_\wedge(\{\mathrm{MDN}\})$
$(q,\Box_a \phi)$	C	$A_\Box(a,\{\mathrm{MPN}(c) \mid c \in C\})$	$\min_C A_\Box(a,\{\mathrm{MDN}\})$

Table 4.11: Selection policy.

Node label	Children	Chosen child
$(q,\neg\phi)$	$\{c\}$	c
$(q,\phi_1 \wedge \phi_2)$	C	$\arg\min_C A_\wedge(\{\mathrm{MDN}\})$
$(q,\Box_a \phi)$	C	$\arg\min_C A_\Box(a,\{\mathrm{MDN}\})$

node as a function of its children are defined in Table 4.10. Finally, the selection procedure base on the effort numbers to decide how to descend the global tree is given in Table 4.11. The stopping condition, Table 4.9, 4.10, and 4.11, as well as Algorithm 8 together define Minimal Proof Search (MPS).

Before studying some theoretical properties of (dis)proofs, minimal (dis)proof numbers, and MPS, let us point out that for any exploration tree, not necessarily produced by MPS, we can associate to each node an MPN and an MDN by using the initialization described in Table 4.9 and the heredity rule described in Table 4.10.

4.4.2 Correctness

The first property we want to prove about MPS is that the descent does not get stuck in a solved subtree.

Proposition 24. *For any internal node n with finite effort numbers, the child c selected by the procedure described in Table 4.11 has finite effort numbers.* $\mathrm{MPN}(n) \neq \infty$ *and* $\mathrm{MDN}(n) \neq \infty$ *imply* $\mathrm{MPN}(c) \neq \infty$ *and* $\mathrm{MDN}(c) \neq \infty$.

Proof. If the formula associated with n has shape $\neg\phi$, then $\mathrm{MDN}(c) = \mathrm{MPN}(n) \neq \infty$ and $\mathrm{MPN}(c) = \mathrm{MDN}(n) \neq \infty$.

If the formula associated with n is a conjunction, then it suffices to note that no child of n has an infinite minimal proof number and at least one child has a finite minimal disproof number, and the result follows from the definition of the selection procedure. This also holds if the formula associated with n is of the form $\Box_a \phi'$. □

As a result, each descent ends in a non solved leaf. Either the associated formula is of the form p and the leaf gets solved, or the leaf becomes an internal node and its children are associated with structurally smaller formulas.

Proposition 25. *The MPS algorithm terminates in a finite number of steps.*

Proof. Let F be the set of lists of formulas ordered by decreasing structural complexity, that is, $F = \{l = (\phi_0, \ldots, \phi_n) | n \in \mathbb{N}, \phi_0 \geq \cdots \geq \phi_n\}$. Note that the lexicographical ordering (based on structural complexity) $<_F$ is wellfounded on F. Recall that there is no infinite descending chains with respect to a well-founded relation.

Consider at some time t the list l_t of formulas associated with the non solved leaves of the tree. Assuming that l_t is ordered by decreasing structural complexity, we have $l_t \in F$. Observe that a step of the algorithm results in a list l_{t+1} smaller than l_t according to the lexicographical ordering and that successive steps of the algorithm result in a descending chain in F. Conclude that the algorithm terminates after a finite number of steps for any input formula ϕ with associated list $l_0 = (\phi)$. □

Since the algorithm terminates, we know that the root of the tree will eventually be labelled with a infinite minimal (dis)proof number and thus will be *solved*. It remains to be shown that this definition of a solved tree coincides with containing (dis)proof starting at the root.

Proposition 26. *If a node n is associated with (q, ϕ), then $\mathrm{MDN}(n) = \infty$ (resp. $\mathrm{MPN}(n) = \infty$) if and only if the tree corresponding to n contains a proof (resp. disproof) that $q \models \phi$ as a subtree with root n.*

Proof. We proceed by structural induction on the shape of trees.

For the base case when n has no children, either $\phi = p$ or ϕ is not atomic. In the first case, n is a terminal node so contains a (dis)proof (n itself) and we obtain the result by definition of MPN nad MDN as per Table 4.9. In the second case, ϕ is not atomic and n has no children so n does not contain a proof nor a disproof. Table 4.9 and Lemma 1 show that the effort numbers are both finite.

For the induction case when $\phi = \neg\phi'$, we know that n has one child c associated to ϕ'. If c contains a proof (resp. disproof) that $q \models \phi'$, then n contains a disproof (resp. proof) that $q \models \phi$. By induction hypothesis, we know that $\mathrm{MPN}(c) = \infty$ (resp. $\mathrm{MDN}(c) = \infty$) therefore, using Table 4.10, we know that $\mathrm{MDN}(n) = \infty$ (resp. $\mathrm{MPN}(n) = \infty$). Conversely if c does not contain a proof nor a disproof, then n does not contain a proof nor a disproof, and we know from the induction hypothesis and Table 4.10 than $\mathrm{MPN}(n) = \mathrm{MDN}(c) < \infty$ and $\mathrm{MDN}(n) = \mathrm{MPN}(c) < \infty$.

The other induction cases are similar but make use of the assumption that aggregating inifinite costs results in infinite costs and that aggregating finite numbers of finite costs results in finite costs. □

Theorem 5. *The MPS algorithm takes a formula ϕ and a state q as arguments and returns after a finite number of steps an exploration tree that contains a (dis)proof that $q \models \phi$.*

4.4.3 Minimality of the (Dis)Proofs

Now that we know that MPS terminates and returns a tree containing a (dis)proof, we need to prove that this (dis)proof is of minimal cost.

The two following propositions can be proved by a simple structural induction on the exploration tree, using Table 4.9 and the admissibility of I and J for the base case and Table 4.10 for the inductive case.

Proposition 27. *If a node n is solved, then the cost of the contained (dis)proof is given by the minimal (dis)proof number of n.*

Proof. Straightforward structural induction on the shape of the tree using the first half of Table 4.9 for the base case and Table 4.10 for the induction step. □

Proposition 28. *If a node n is associated with (q, ϕ), then for any proof m (resp. disproof) that $q \models \phi$, we have $\mathrm{MPN}(n) \leq K(m)$ (resp. $\mathrm{MDN}(n) \leq K(m)$).*

Proof. Structural induction on the shape of the tree, using the second half of Table 4.9 and the admissibility of I and J (Prop. 23) for the base case and Table 4.10 for the inductive case. □

Since the aggregators for the cost function are increasing functions, then $\mathrm{MPN}(n)$ and $\mathrm{MDN}(n)$ are non decreasing as we add more nodes to the tree n.

Proposition 29. *For each disproved internal node n in a tree returned by the MPS algorithm, at least one of the children of n minimizing the MDN is disproved.*

Sketch. If we only increase the minimal (dis)proof number of a leaf, then for each ancestor, at least one of either the minimal proof number of the minimal disproof number remains constant.

Take a disproved internal node n, and assume we used the selection procedure described in Table 4.11. On the iteration that lead to n being solved, the child c of n selected was minimizing the MDN and this number remained constant since $\mathrm{MPN}(c)$ raised from a finite value to ∞.

Since the MDN of the siblings of c have not changed, then c is still minimizing the MDN after it is solved. □

Theorem 6. *The tree returned by the MPS algorithm contains a (dis)proof of minimal cost.*

4.4.4 Optimality

The MPS algorithm is not optimal in the most general sense because it is possible to have better algorithm in some cases by using transpositions, domain knowledge, or logical reasoning on the formula to be satisfied.

For instance, take $\phi_1 = \Diamond_a(p \wedge \neg p)$ and ϕ_2 some non trivial formula satisfied in a state q. If we run the MPS algorithm to prove that $q \models \phi_1 \vee \phi_2$, it will explore at least a little the possibility of proving $q \models \phi_1$ before finding the minimal proof through ϕ_2. We can imagine that a more "clever" algorithm would recognize

that ϕ_1 is never satisfiable and would directly find the minimal proof through ϕ_2.

Another possibility to outperform MPS is to make use of transpositions to shortcut some computations. MPS indeed explores structures according to the MMLK formula shape, and it is well-known in modal logic that bisimilar structures cannot be distinguished by MMLK formulas. It is possible to express an algorithm similar to MPS that would take transpositions into account, adapting ideas from PNS [139, 100, 73]. We chose not to do so in this article for simplicity reasons.

Still, MPS can be considered optimal among the programs that do not use reasoning on the formula itself, transpositions or domain knowledge. Stating and proving this property formally is not conceptually hard, but we have not been able to find simple definitions and a short proof that would not submerge the reader with technicalities. Therefore we decided only to describe the main ideas of the argument from a high-level perspective.

Definition 26. A pair (q', ϕ') is *similar* to a pair (q, ϕ) with respect to an exploration tree n associated with (q, ϕ) if q' can substitute for q and ϕ' for ϕ in n.

Let n associated with (q, ϕ) be an exploration tree with a finite MPN (resp. MDN), then we can construct a pair (q', ϕ') similar to (q, ϕ) with respect to n such that there is a proof that $q' \models \phi'$ of cost exactly $\mathrm{MPN}(n)$ (resp. a disproof of cost $\mathrm{MDN}(n)$).

Definition 27. An algorithm A is *purely exploratory* if the following holds. Call n the tree returned by A when run on a pair (q, ϕ). For every pair (q', ϕ') similar to (q, ϕ) with respect to n, running A on (q', ϕ') returns a tree structurally equivalent to n.

Depth first search, if we were to return the explored tree, and MPS are both examples of purely exploratory algorithms.

Proposition 30. *If a purely exploratory algorithm A is run on a problem (q, ϕ) and returns a solved exploration tree n where $\mathrm{MPN}(n)$ (resp. $\mathrm{MDN}(n)$) is smaller than the cost of the contained proof (resp. disproof), then we can construct a problem (q', ϕ') similar with respect to n such that A will return a structurally equivalent tree with the same proof (resp. disproof) while there exists a proof of cost $\mathrm{MPN}(n)$ (resp. disproof of cost $\mathrm{MDN}(n)$).*

Note that if the cost of a solved exploration tree n is equal to its MPN (resp. MDN), then we can make MPS construct a solved shared root subtree of n just by influencing the tie-breaking in the selection policy described in Table 4.11.

Theorem 7. *If a purely exploratory algorithm A returns a solved exploration tree n, either this tree (or a subtree) can be generated by MPS or A is not guaranteed to return a tree containing a (dis)proof of minimal cost on all possible inputs.*

4.5 Sequential solution concepts in MMLK

We now proceed to define several classes of formulas to express interesting properties about games.

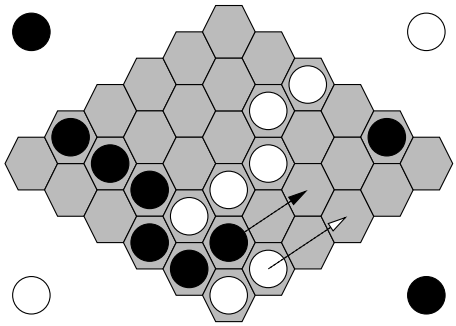
We will assume for the remainder of the paper that one distinguished player is denoted by A and the other players (if any) are denoted by B (or B^1, ..., B^k). Assume a distinguished atomic propositions w, understood as a label of final positions won by A. We also use a variant of the *normal play* assumption, namely, when a position s is won by A, no other player has a legal move in s. When a position s is lost by A, then A has no legal moves in s but other players have a *pass* move that loop to s. That is, for every state s lost by A, we have $s \xrightarrow{B} s$.

Reachability A natural question that arises in one-player games is *reachability*. In this setting, we are not interested in reaching a specific state, but rather in reaching any state satisfying a given property.

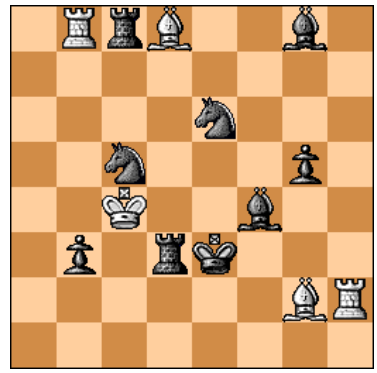
Definition 28. We say that a player A can *reach* a state satisfying ϕ from a state q in exactly n steps if $q \models \underbrace{\Diamond_A \dots \Diamond_A}_{n \text{ times}} \phi$.

Winning strategy The concept of winning strategies in a finite number of moves in an alternating two-player game can also be represented as as a formula.

Definition 29. Player A has a *winning strategy* of depth less or equal to $2n+1$ in state q if $q \models \omega_{n+1}$, where $\omega_1 = w \vee \Diamond_A w$ and $\omega_n = w \vee \Diamond_A \Box_B \omega_{n-1}$.



(a) HEX position featuring a *ladder* for *Black*.



(b) CHESS position featuring a *helpmate* for *Black* in four moves.

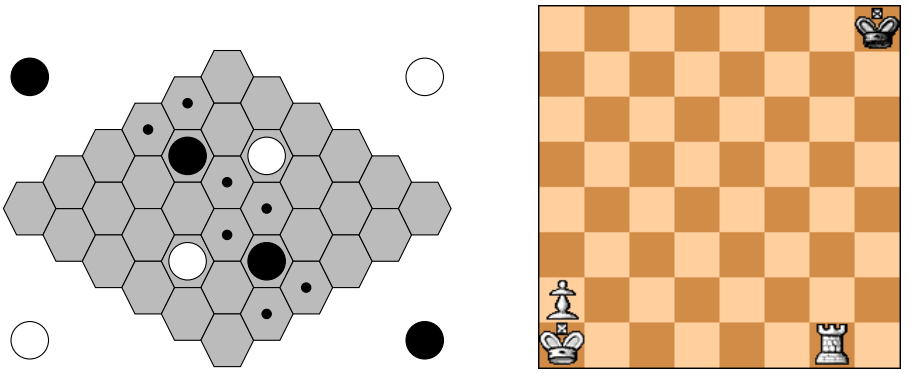
Figure 4.1: Positions illustrating the solution concepts of *ladder* and *helpmate*.

Ladders The concept of *ladder* occurs in several games, particularly HEX and GO [102]. A threatening move for player A is a move such that, if it was possible for A to play a second time in a row, then A could win. A ladder is a sequence of threatening moves by A followed by defending moves by B, ending with A fulfilling their objective.

Definition 30. Player A has a *ladder* of depth less or equal to $2n+1$ in state s if $q \models \mathrm{L}_{2n+1}$, where $\mathrm{L}_1 = w \vee \Diamond_A w$ and $\mathrm{L}_{2n+1} = w \vee \Diamond_A((w \vee \Diamond_A w) \wedge \Box_B \mathrm{L}_{2n-1})$.

For instance, Figure 4.1a presents a position of the game HEX where the goal for each player is to connect their border by putting stones of their color. In this position, *Black* can play a successful ladder thereby connecting the left group to the bottom right border.

Helpmates In a CHESS *helpmate*, the situation seems vastly favourable to player *Black*, but the problemist must find a way to have the Black king checkmated. Both players move towards this end, so it can be seen as a cooperative game. *Black* usually starts in helpmate studies. See Figure 4.1b for an example. A helpmate in at most $2n$ plies can be represented through the formula H_n where $\mathrm{H}_0 = w$ and $\mathrm{H}_n = w \vee \Diamond_B \Diamond_A \mathrm{H}_{n-1}$.



(a) HEX position featuring a *progress tree* of depth 7 for *Black*.

(b) CHESS endgame featuring a *progress tree* of depth 11 for *White*.

Figure 4.2: Positions illustrating the concept of *progress tree*.

Selfmates A *selfmate*, on the other hand, is a situation where *Black* forces *White* to checkmate the Black King, while *White* must do their best to avoid this. *Black* starts moving in a selfmate and a position with a selfmate satifies S_n for some n, where $\mathrm{S}_0 = w$ and $\mathrm{S}_n = w \vee \Diamond_B \Box_A \mathrm{S}_{n-1}$.

Progress Trees It occurs in many two-player games that at some point near the end of the game, one player has a winning sequence of n moves that is relatively independent of the opponent's moves. For instance Figure 4.2 presents a HEX position won for *Black* and a CHESS position won for *White*. In both cases, the opponent's moves cannot even delay the end of the game.

To capture this intuition, we define a new solution concept we name *progress tree*. The idea giving its name to the concept of progress trees is that we want the player to focus on those moves that brings them closer to a winning state, and discard the moves that are out of the winning path.

Definition 31. Player A has a *progress tree* of depth $2n + 1$ in a state q if $q \models \mathrm{PT}_{n+1}$, where $\mathrm{PT}_1 = w \vee \Diamond_A w$ and $\mathrm{PT}_n = w \vee \Diamond_A(\pi_{n-1} \wedge \Box_B \mathrm{PT}_{n-1})$.

We can check states for progress trees using any of the model checking algorithms presented earlier, effectively giving rise to four new specialized

Table 4.12: Search statistics for a DFPS on positions along a principal variation of the CHESS problem in Figure 4.2b.

Model checking problem	Time (s)	Number of queries		
		`atomic`	`listmoves`	`play`
PT_3	0.1	6040	328	5897
ω_3	0.2	11172	624	5587
PT_4	1.4	99269	5312	98696
ω_4	3.5	194429	10621	97217
PT_5	23.6	1674454	88047	1668752
ω_5	63.8	3382102	181442	1691055
PT_6	260.4	25183612	1297975	25106324
ω_6	953.6	52209939	2759895	26104986

algorithms. Note that if a player has a progress tree of depth $2n + 1$ in some state, then they also have a winning strategy of depth $2n + 1$ from that state (see Proposition 31). Therefore, if we prove that a player has a progress tree in some position, then we can deduce that they have a winning strategy.

We tested a naive implementation of the DFPS model checking algorithms on the position in Figure 4.2b to check for progress trees and winning strategies. The principal variations consists for *White* in moving the pawn up to the last row and move the resulting queen to the bottom-right hand corner to deliver checkmate. To study how the solving difficulty increases with respect to the size of the formula to be checked, we model checked every position on a principal variation and present the results in Table 4.12.

We can see that proving that a progress tree exists becomes significantly faster than proving an arbitrary winning strategy as the size of the problem increases. We can also notice that the overhead of checking for a path at each α node of the search is more than compensated by the early pruning of moves not contributing to the winning strategy.

4.6 Understanding game tree algorithms

We now turn to show how the MMLK Model Checking framework can be used to develop new research in game tree search. As such, the goal of this section is not to put forward a single well performing algorithm, nor to prove difficult theorems with elaborate proofs, but rather to demonstrate that the MMLK Model Checking is an appropriate tool for designing and reasoning about new game tree search algorithms.

By defining appropriate formulas classes, we can simulate many existing algorithms by solving model checking problems in MMLK with specific search algorithms.

Definition 32. Let ϕ be a formula, S be a model checking algorithm and A be a specific game algorithm. We say that (ϕ, S) *simulates* A if for every game, for every state q where A can be applied, we have the following: solving $q \models \phi$ with S will explore exactly the same states in the same order and return the same result as algorithm A applied to initial state q.

Table 4.13 presents how combining the formulas defined later in this section with model checking algorithms for MMLK allows to simulate many important algorithms. We use the model checking algorithms defined in Section 4.3, DFPS, PNPS, and MCPS as well as MPS proposed by Saffidine [119]. For instance, using the DFPS algorithm to model-check an APS_n formula on a HEX position represented as a state of a GA is exactly the same as running the Abstract Proof Search algorithm on that position.

4.6.1 One-player games

Many one-player games, the so-called puzzles, involve finding a path to a terminal state. Ideally this path should be the shortest possible. Examples of such puzzles include the 15-PUZZLE and RUBIK'S CUBE.

Recall that we defined a class of formulas for reachability in exactly n steps in Definition 28. Similarly we define now a class of formulas representing the existence of a path to a winning terminal state within n moves.

Definition 33. We say that agent A has a *winning path* from a state q if q satisfies π_n where π_n is defined as $\pi_0 = w$ and $\pi_n = w \vee \Diamond_A \pi_{n-1}$ if $n > 0$.

Table 4.13: Different algorithms expressed as a combination of a formula class and a search paradigm.

Formula	Model Checking Algorithm			
	DFPS	PNPS	MCPS	MPS
π_n	Depth-first search	Greedy BFS [111]	Single-player MCTS [132]	A* [60]
ω_n	$\alpha\beta$ [75]	PNS [4]	MCTS solver [165]	DFPN+ [103]
PA_n	Paranoid [149]	Paranoid PNS [129]	Multi-player MCTS [105]	
$\lambda_{d,n}$	Lambda search [153]	Lambda PNS [169][1]		
β_n	Best reply search [130]		MCTS-BRS [106]	
APS_n	Abstract proof search [23]			

[1] We actually need to change the update rule for the p in internal $\phi_1 \wedge \phi_2$ nodes in PNPS from $\sum_C p$ to $\max_C p$.

4.6.2 Two-player games

We already defined the *winning strategy* formulas ω_n in Definition 29. We will now express a few other interesting formulas that can be satisfied in states in two player games.

λ-Trees λ-trees have been introduced [153] as a generalisation of ladders as seen in Section 4.5. We will refrain from describing the intuition behind λ-trees here and will be satisfied with giving the formal corresponding property as they only constitute an example of the applicability of our framework.

Definition 34. A state q has an *λ-tree* of order d and maximal depth n for player A if $q \models \lambda_{d,n}$, where $\lambda_{0,n} = \lambda_{d,0} = w$, $\lambda_{d,1} = w \vee \Diamond_A w$, and $\lambda_{d,n} = w \vee \Diamond_A(\lambda_{d-1,n-1} \wedge \Box_B \lambda_{d,n-2})$.

λ-trees are a generalisation of ladders as defined in Definition 30 since a ladder is a λ-tree of order $d = 1$.

Abstract proof trees Abstract proof trees were introduced to address some perceived practical limitations of $\alpha\beta$ when facing a huge number of moves. They have been used to solve capture problems for the game of GO. We limit ourselves here to describing how we can specify in MMLK that a state is root to an an abstract proof tree. Again, we refer the reader to the literature for the intuition about abstract proof trees and their original definition [23].

Definition 35. A state q has an *abstract proof tree* of order n for player A if $q \models \mathrm{APS}_n$, where $\mathrm{APS}_0 = w$, $\mathrm{APS}_1 = w \vee \Diamond_A w$, and $\mathrm{APS}_n = w \vee \Diamond_A(\mathrm{APS}_{n-1} \wedge \Box_B \mathrm{APS}_{n-1})$.

Other concepts Many other interesting concepts can be similarly implemented via a class of appropriate formulas. Notably minimax search with iterative deepening, the Null-move assumption, and Dual Lambda-search [145] can be related to model checking some MMLK formulas with DFPS.

4.6.3 Multiplayer games

Paranoid Algorithm The Paranoid Hypothesis was developed to allow for $\alpha\beta$ style safe pruning in multiplayer games [149]. It transforms the original

$k+1$-player game into a two-player game A versus B. In the new game, the player B takes the place of $B^1, \ldots, B^k$ and B is trying to prevent player A from reaching a won position. Assuming the original turn order is fixed and is $A, B^1, \ldots, B^k, A, B^1, \ldots$, we can reproduce a similar idea in MMLK.

Definition 36. Player A has a *paranoid win* of depth $(k+1)n$ in a state q if $q \models \mathrm{PA}_{n,0}$, where $\mathrm{PA}_{n,0}$ is defined as follows.

$$\begin{aligned} \mathrm{PA}_{0,i} &= w \\ \mathrm{PA}_{n,0} &= w \vee \Diamond_A \mathrm{PA}_{n-1,1} \\ \mathrm{PA}_{n,i} &= \Box_{B^i} \mathrm{PA}_{n-1,i+1} \text{ for } 1 \leq i < k \\ \mathrm{PA}_{n,k} &= \Box_{B^k} \mathrm{PA}_{n-1,0} \end{aligned} \tag{4.1}$$

Observe that in a $k+1$-player game, if $0 \leq j < k$ then $\mathrm{PA}_{(k+1)n+j,j}$ can be expressed as $\Box_{B^j} \Box_{B^{j+1}} \ldots \Box_{B^k} \mathrm{PA}_{(k+1)n,0}$

Best Reply Search Best Reply Search (BRS) is a new search algorithm for multiplayer games [130]. It consists of performing a minimax search where only one opponent is allowed to play after A. For instance a principal variation in a BRS search with $k = 3$ opponents could involve the following turn order A, B_2, A, B_1, A, B_1, A, B_3, A, ... instead of the regular A, B_1, B_2, B_3, A, B_1, B_2, B_3, ...

The rationale behind BRS is that the number of moves studied for the player in turn in any variation should only depend on the depth of the search and not on the number of opponents. This leads to an artificial player selecting moves exhibiting longer term planning. This performs well in games where skipping a move does not influence the global position too much, such as CHINESE CHECKERS.

Definition 37. Player A has a *best-reply search win* of depth $2n+1$ in a state q if $q \models \beta_n$, where $\beta_1 = w \vee \Diamond_A w$ and $\beta_n = w \vee \Diamond_A \bigwedge_{i=1}^{k} \Box_{B^i} \beta_{n-1}$.

4.6.4 Expressing properties of the algorithms

We now demonstrate that using the MMLK model checking framework for game tree search makes some formal reasoning straightforward. Again, the goal of

this section is not to demonstrate strong theorems with elaborate proofs but rather show that the framework is convenient for expressing certain properties and helps reasoning on them.

It is easy to prove by induction on the depth that lambda trees, abstract proof trees, and progress trees all have winning strategies as logical consequence.

Proposition 31. *For all order d and depth n, we have $\models \lambda_{d,2n+1} \to \omega_{n+1}$, $\models \mathrm{APS}_n \to \omega_n$, and $\models \mathrm{PT}_n \to \omega_n$.*

Proof. We prove the implication between λ-trees and winning strategy by induction on the depth n. The proofs for abstract proof trees and progress trees are similar and are omitted.

Base case $n = 0$. If the depth is $n = 0$ then we have $\lambda_{d,1} = w \vee \Diamond_A w$ and $\omega_1 = w \vee \Diamond_A w$ so the property holds.

Induction case, assuming $\models \lambda_{d,2n+1} \to \omega_{n+1}$, let us show that $\models \lambda_{d,2n+3} \to \omega_{n+2}$. If $d = 0$, then $\lambda_{d,2n+3} = w$ so the property holds. Else, $d > 0$ and we have $\lambda_{d,2n+3} = w \vee \Diamond_A(\lambda_{d-1,2n+2} \wedge \Box_B \lambda_{d,2n+1})$. By induction hypothesis we obtain $\models \lambda_{d,2n+3} \to w \vee \Diamond_A(\lambda_{d-1,2n+2} \wedge \Box_B \omega_{n+1})$. By weakening we have $\models \lambda_{d,2n+3} \to w \vee \Diamond_A \Box_B \omega_{n+1}$, that is $\models \lambda_{d,2n+3} \to w \vee \omega_{n+2}$. □

Therefore, whenever we succeed in proving that a position features, say, an abstract proof tree, then we know it also has a winning strategy for the same player: for any state q, $q \models \mathrm{APS}_n$ implies $q \models \omega_n$.

On the other hand, in many games, it is possible to have a position featuring a winning strategy but no lambda tree, abstract proof tree, or even progress tree. Before studying the other direction further, we need to rule out games featuring *zugzwangs*, that is, positions in which a player would rather pass and let an opponent make the next move.

Definition 38. A *ϕ-zugzwang* for player A against players $B^1, \ldots, B^k$ is a state q such that $q \models \neg\phi \wedge (\bigvee_i \Box_{B^i} \phi)$. A game is *zugzwang-free* for a set of formulas Φ and player A against players $B^1, \ldots, B^k$ if for every state q, and every formula $\phi \in \Phi$, q is not a ϕ-zugzwang for A against $B^1, \ldots, B^k$.

We denote the set games that are zugzwang-free for Φ as $Z(\Phi)$. A formula ψ is valid in zugzwang-free games for Φ, if for any game G in $Z(\Phi)$ and any state s, we have $G, s \models \psi$. In such case we write $\models_{Z(\Phi)} \psi$.

We can use this definition to show that in games zugzwang-free for winning strategies, such as CONNECT-6 or HEX, an abstract proof tree and a progress tree are equivalent to a winning strategy of the same depth.

Proposition 32. *Consider a two-player game zugzwang-free for winning strategies. For any depth n and any state q, $q \models \omega_n$ implies $q \models \mathrm{APS}_n$ and $q \models \omega_n$ implies $q \models \mathrm{PT}_n$. That is $\models_{Z(\Phi)} \omega_n \to \mathrm{APS}_n$ and $\models_{Z(\Phi)} \omega_n \to \mathrm{PT}_n$ where $\Phi = \{\omega_i, i \in \mathbb{N}\}$.*

Proof. We prove the result involving abstract proof trees and winning strategies by induction on n. The other result can obtained with a similar proof. Assume a zugzwang-free game for winning strategies. We want to prove that for any state q in the game, if $q \models \omega_n$ then we have $q \models \mathrm{APS}_n$

Since $\omega_0 = \mathrm{APS}_0 = w$, the property holds for the base case, $n = 0$.

Assume that for all q, $q \models \omega_n$ implies $q \models \mathrm{APS}_n$, and take q such that $q \models \omega_{n+1}$. Since $\omega_{n+1} = w \vee \Diamond_A \Box_B \omega_n$ then either $q \models w$ in which case $q \models \mathrm{APS}_{n+1}$ or we can find q' such that $q' \models \Box_B WS$ and $q \xrightarrow{A} q'$. In this case, it remains to prove that $q' \models \mathrm{APS}_n$ and $q' \models \Box_B \mathrm{APS}_n$.

To show that $q' \models \Box_B \mathrm{APS}_n$, consider q'' such that $q' \xrightarrow{B} q''$. We know that $q' \models \Box_B \omega_n$ so $q'' \models \omega_n$ and by induction hypothesis, we have $q'' \models \mathrm{APS}_n$.

To show that $q' \models \mathrm{APS}_n$, recall that the game is zugzwang-free for winning strategies, and in particular, it is zugzwang-free for ω_n. It means that there is no state $\hat{q}$ such that $\hat{q} \models \neg\omega_n \wedge \Box_B \omega_n$. By taking $\hat{q}$ to be q', we have $q' \models \omega_n \vee \neg \Box_B \omega_n$. Since we know that $q' \models \Box_B \omega_n$ we derive that $q' \models \omega_n$. The induction hypothesis allows us to conclude. □

The usual understanding of zugzwang is in two player games with ϕ a winning strategy formula or a formula representing forcing some material gain in CHESS. The more general definition we have given allows for multiplayer games and other solution concepts besides winning strategies. For instance, it is possible to show that best reply wins are more common than paranoid wins in games which are zugzwang-free for paranoid wins.

Proposition 33. *Consider a multiplayer game with $k+1$ players, zugzwang-free for paranoid wins. For any depth n and any state q, $q \models \mathrm{PA}_{(k+1)n,0}$ implies $q \models \beta_n$. That is $\models_{Z(\Phi)} \mathrm{PA}_{(k+1)n,0} \to \beta_n$ where $\Phi = \{\mathrm{PA}_{i,j}, i \in \mathbb{N}, j \in \mathbb{N}\}$.*

As a result, in such a game, if a search for a best reply win fails, we know there is no need to carry a search for a paranoid win. Since looking for a best reply win of depth $2n + 1$ is usually much faster than looking for a paranoid win of depth $(k + 1)n$, this formal result can be seen as a partial explanation of the experimental success of Best Reply Search in CHINESE CHECKERS [130].

4.6.5 Examining new combinations

We have seen that we could obtain previously known algorithms by combining model checking algorithms with solution concepts. On the one hand, some solution concepts such winning path and winning strategy, were combined with the four possible search paradigms in previous work. On the other hand, other solution concepts such as abstract proof trees were only investigated within the depth-first paradigm.

It is perfectly possible to model check a paranoid win using the MCPS algorithm, for instance, leading to a new Minimal Paranoid Win Search algorithm. Similarly model checking abstract proof trees with PNPS would lead to a new Proof Number based Abstract Proof Search (PNAPS) algorithm. Preliminary experiments in HEX without any specific domain knowledge added seem to indicate that PNAPS does not seem to perform as well as Abstract Proof Search, though.

Finally, most of the empty cells in Table 4.13 can be considered as new algorithms waiting for an optimized implementation and a careful evaluation.

4.7 Related work and discussion

In CTL model checking, finding a minimal witness or a minimal counterexample is NP-complete [31]. MMLK model checking, on the contrary, though PTIME-complete [84], allows finding minimal witnesses/counterexamples relatively efficiently as we shall see in this chapter.

The tableau-based model checking approach by Cleaveland for the μ-calculus seems to be similar to ours [34], however it would need to be adapted to handle (dis)proof cost. Also, in our understanding, the proof procedure *check1* presented by Cleaveland can be seen as an unguided Depth First Search (DFS) while our approach is guided towards regions of minimal cost.

The two algorithms most closely related to MPS are AO*, a generalization of A* to And/Or trees, and DFPN+ [103], a variant of DFPN, itself a depth-first variant of PNS [4].

DFPN+ is typically only used to find a winning strategy for either player in two-player games. MPS, on the contrary, can be applied to solve other interesting problems without a cumbersome And/Or graph prior conversion. Example of such problems range from finding ladders in two-player games to finiding paranoid wins in multi-player games. Another improvement over DFPN+ is that we allow for a variety of (dis)proof size definitions. While DFPN+ is set to minimize the total edge cost in the proof, we can imagine minimizing, say, the number of leaves or the depth of the (dis)proof.

Besides the ease of modelling allowed by MMLK rather than And/Or graphs, another advantage of MPS over AO* is that if the problem is not solvable, then MPS finds a minimal disproof while AO* does not provide such a guarantee.

In his thesis, Nagai derived the DFPN algorithm from the equivalent best-first algorithm PNS [103]. Similarly, we can obtain a depth-first version of MPS from the best first search version presented here by adapting Nagai's transformation. Such a depth-first version should probably be favoured in practice, however we decided to present the best first version in this article for two main reasons. We believe the best-first search presentation is more accessible to the non-specialists. The proofs seemed to be easier to work through in the chosen setting, and they can later be extended to the depth-first setting.

Another trend of related work is connecting modal logic and game theory [157, 167, 81]. In this area, the focus is on the concept of Nash equilibria, extensive form games, and coalition formation. As a result, more powerful logic than the restricted MMLK are used [6, 159, 58]. Studying how the model checking algorithms presented in this chapter can be extended for these settings is an interesting path for future work.

We have shown that the Multi-agent Modal Logic K was a convenient tool to express various kind of threats in a game independent way. Victor Allis provided one of the earliest study of the concept of threats in his *Threat space search* algorithm used to solve GOMOKU [5].

Previous work by Schaeffer et al. was also concerned with providing a unifying view of heuristic search and the optimization tricks that appeared in both single-agent search and two-player game search [134].

The model used in this chapter differs from the one used in GGP called Multi-Agent Environment (MAE) [138]. In an MAE, a transition correspond to a joint-action. That is, each player decide a move simultaneously and the combination of these moves determines the next state. In a GA, as used in this chapter, the moves are always sequential. It is possible to simulate sequential moves in an MAE by using *pass* moves for the non acting agents, however this ties the turn player into the game representation. As a result, testing for solution concepts where the player to move in a given position is variable is not possible with an MAE. For instance, it is not possible to formally test for the existence of a ladder in a GGP representation of the game of GO because we need to compute the successors of a given position after a white move and alternatively after a black move.

We envision a depth-first adaptation of MPS similar to Nagai's transformation of PNS into DFPN. Alternatively, we can draw inspiration from PN^2 [4] and replace the heuristic functions I and J by a nested call to MPS, leading to an MPS^2 algorithm trading time for memory. These two alternative algorithms would directly inherit the correctness and minimality theorems for MPS. The optimality theorem would also transpose in the depth-first case, but it would not be completely satisfactory. Indeed, even though the explored tree will still be admissibly minimal, several nodes inside the tree will have been forgotten and re-expanded multiple times. This trade-off is reminiscent of the one between A* and its depth-first variation IDA* [78].

Representing problems with unit edge costs is already possible within the framework presented in Section 4.2.5. It is not hard to adapt MPS to the more general case as we just need to replace the agent labels on the transitions with (agent, cost) labels. This more general perspective was not developed in this chapter because the notation would be heavier while it would not add much to the intuition and the general understanding of the ideas behind MPS.

Effective handling of transpositions is another interesting topic for future work. It is already nontrivial in PNS [73] and MCTS [125], but it is an even richer subject in this model checking setting as we might want to prove different facts about a given position in the same search. Finding minimal (dis)proofs is more challenging when taking transpositions into account because of the double count problem. While it is possible to obtain a correct algorithm returning minimal (dis)proofs by using functions based on propagating sets of individual costs

instead of real values in Section 4.2.5, similarly to previous work in PNS [100], such a solution would hardly be efficient in practice and would not necessarily be optimal. The existing literature on PNS and transpositions can certainly be helpful in addressing efficient handling of transpositions in MMLK model checking [139, 100, 73].

4.8 Conclusion

We have seen the MMLK was an appropriate framework for formal definitions of solution concepts for perfect information games. We have shown that research on game tree search could be a source of inspiration when designing algorithms to solve the model checking problem for MMLK. Also, combining modal formulas and a model checking algorithms yields a variety of game tree search algorithms that can be modeled in the same framework. This makes it easy to test known algorithms as well to define new ones. Finally, non-trivial properties of game search algorithms can be proved in the modal logic formalism with just a few formula manipulations.

Table 4.13 reveals many interesting previously untested possible combinations of formula classes and search algorithms. Implementing and optimizing one specific new combination for a particular game could lead to insightful practical results. For instance, it is quite possible that a Proof Number version of Abstract Proof Search would be successful in the capture game of GO [23].

We have also presented Minimal Proof Search (MPS), a model checking algorithm for MMLK. MPS has been proven correct, and it has been proved that the (dis)proof returned by MPS was minimizing a generic cost function. The only assumption about the cost function is that it is defined recursively using increasing aggregators. Finally, we have shown that MPS was optimal among the purely exploratory model checking algorithms for MMLK.

Nevertheless, the proposed approach has a few limitations. MPS is a best first search algorithm and is memory intensive; the cost functions addressed in the chapter cannot represent variable edge cost; and MPS cannot make use of transpositions in its present form. Still, we think that these limitations can be overcome in future work.

Beside evaluating and improving the practical performance of MPS, future work can also study to which extent the ideas presented in this chapter can be

applied to model checking problems in more elaborate modal logics and remain tractable.

5 Games with Simultaneous Moves

This chapter defines and focuses on stacked-matrix games*, that is two-player zero-sum games featuring simultaneous moves. Alpha-beta pruning can be generalized to stacked-matrix games, however computing the alpha and beta bounds is costly as it involves solving Linear Programs (LPs). We develop a few heuristical optimizations that allow to mitigate the time spent solving LPs, eventually leading to an algorithm solving* GOOFSPIEL *faster than backward induction and sequence form solving.*

The stacked-matrix games formalism can also model the combat phase of Real-Time Strategy (RTS) games. However the time constraints associated to building an Artificial Intelligence (AI) for an RTS game are so tight that practical settings cannot be solved exactly. On the other hand, we show that approximate heuristical search is possible and leads to much better performance than existing script-based approaches.

This Chapter includes results from the following papers.

[127] Abdallah Saffidine, Hilmar Finnsson, and Michael Buro. Alpha-beta pruning for games with simultaneous moves. In Hoffmann and Selman [66], pages 556–562

[30] David Churchill, Abdallah Saffidine, and Michael Buro. Fast heuristic search for RTS game combat scenarios. In Mark Riedl and Gita Sukthankar, editors, 8th AAAI Conference on Artificial Intelligence and Interactive Digital Entertainment (AIIDE)*, pages 112–117, Palo Alto, California, USA, October 2012. AAAI Press*

Contents

5.1 Stacked-matrix games

While search-based planning approaches have had a long tradition in the construction of strong AI systems for abstract games like CHESS and GO, only in recent years have they been applied to modern video games, such as First-Person Shooter (FPS) and RTS games [108, 29].

In this chapter, we study two-player zero-sum games featuring simultaneous moves called *stacked-matrix games*. We first show how it relates to the more general class of MAE and that combat situations in RTS games can be seen as

stacked-matrix games. We then provide with an algorithm to perform safe pruning in a depth-first search in this class of games, thus generalizing alpha-beta pruning. Using this algorithm, we are able to solve GOOFSPIEL instances more efficiently. In RTS games, the time constraints are tight and the goal is to find a good move rather than to determine the value of the game. We therefore show how efficient approximate search can be performed on stacked-matrix games, focussing on RTS combat settings.

Classical abstract games such as CHESS or GO are purely sequential zero-sum two-player games. To model some other games such as CHINESE CHECKERS it is necessary to drop the two-player assumption. In this chapter, we study the class of games obtained by dropping the pure sequentiality assumption.

5.1.1 Formal model

Definition 39. A *stacked-matrix game* is a transition system $\langle S, R, \rightarrow, L, \lambda \rangle$ in which the following restriction holds:

- The set of transition labels can be seen as the cartesian square of a set of *moves* M, $R = M \times M$;
- The state labels are bounded real numbers, $L \subseteq [b, a]$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Stacked-matrix games are two-player games with simultaneous moves. The interpretation of the transition relation is that when in a state s, both players choose a move, say m_1 and m_2 and the resulting state s' is obtained as a combination of s and the joint move (m_1, m_2): $s \xrightarrow{m_1,m_2} s'$.

A state is *final*, if it has no outgoing transitions. The set of *final states* is $F = \{s \in S, \neg\exists s' \in S, (m_1, m_2) \in M \times M, s \xrightarrow{m_1,m_2} s'\}$. States that are not final are called *internal*.

Definition 40. The *score* of a final state $s \in F$, $\sigma(s)$, is defined as the maximum outcome if any outcome appears in s, and b otherwise. $\sigma(s) = \max \lambda(s)$ if $\lambda(s) \neq \emptyset$, and $\sigma(s) = b$ if $\lambda(s) = \emptyset$.

To simplify exposure, we assume that transitions are deterministic rather than stochastic or non-deterministic. That is, for every triple (s, m_1, m_2) there

Table 5.1: Definition of the transition function $\rightarrow$ from the game presented in Figure 5.1.

$S \setminus F$	$M \times M$			
	$(1,1)$	$(1,2)$	$(2,1)$	$(2,2)$
s_1	s_3	f_6	f_2	s_2
s_2	s_3	f_6	f_0	f_2
s_3	f_{10}	f_0	f_0	f_{10}

is at most a single state s' such playing (m_1, m_2) in s can lead to s'.

$$\forall s, s', s'' \in S, \forall m_1, m_2 \in M, s \xrightarrow{m_1,m_2} s' \wedge s \xrightarrow{m_1,m_2} s'' \Rightarrow s' = s'' \tag{5.1}$$

Definition 41. A move m_1 is *legal* for *Max* in a state s if there is a transition from s that involves m_1, that is, if there is a move m_2 and a state s' such that $s \xrightarrow{m_1,m_2} s'$. Similarly, we define legal moves for *Min*. The set of legal moves for *Max* and *Min* in a state s are denoted $L_1(s)$ and $L_2(s)$ respectively.

Remark 7. *Let s a state, we want every combination of legal moves to lead to a valid state. To ensure this is the case, we add new final state f_b with score b to the game. We then extend the transition relation so that every missing combination of legal moves from s now leads to f_b.*

Example 11. Consider the following game. $G = \langle S, M \times M, \rightarrow, L, \lambda \rangle$, where $S = \{s_1, s_2, s_3, f_0, f_2, f_6, f_{10}\}$, $M = \{1, 2\}$, $\rightarrow$ is defined as presented in Table 5.1, $L = [0, 10]$, and $\lambda(f_i) = \{i\}$. It is possible to represent G graphically as in Figure 5.1. The first player *Max* performs the row selection and the second player *Min* corresponds to columns.

We will assume in the rest of this chapter that the transition relation forms a Direct Acyclic Graph (DAG).

5.1.2 Related game classes

Multi-Agent Environments (MAEs) formally describe discrete and deterministic multi-agent domains [138]. A MAE can be seen as a transition system where transitions are associated with joint actions executed by the participating agents. It is useful to classify MAEs along several dimensions:

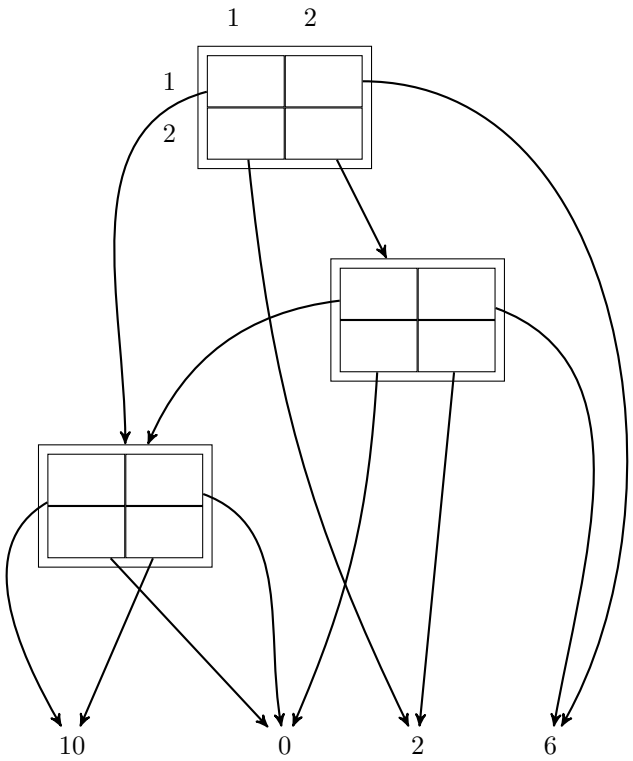


Figure 5.1: Graphical representation of the game from Example 11. The states are from top to bottom and from left to right s_1, s_2, s_3, f_{10}, f_0, f_2, and f_6.

Definition 42. An MAE is said to be

(purely) sequential if in any state, there is at most one agent with more than one legal action;

zero-sum if the sum of utilities of all agents is constant in all final states;

single-player if there is at most one agent, ***two-player*** if there are at most two agents, and ***multiplayer*** otherwise.

Proposition 34. *Stacked-matrix games are equivalent to two-player zero-sum MAE.*

Clearly, the stacked-matrix games constitute a super-class of the multi outcome games in Chapter 3. They can also be seen as the deterministic non-loopy subclass of recursive games [45, 10]. This class of games encompasses a small portion of games appearing in the GGP competition such as BIDDING-TICTACTOE. Furthermore, particular instances of this game class have been studied in [21, 79, 52]. This class encompasses a few card games such as GOOFSPIEL and the two-player version of 6 NIMMT. It can also be used to model the combat phase in RTS games as we shall see.

As a subset of general zero-sum imperfect information games, stacked matrix games can be solved by general techniques such as creating a single-matrix game in which individual moves represent pure strategies in the original game. However, because this transformation leads to an exponential blowup, it can only be applied to tiny problems. In their landmark paper, [77] define the sequence form game representation which avoids redundancies present in above game transformation and reduces the game value computation time to polynomial in the game tree size. In the experimental section we present data showing that even for small stacked matrix games, the sequence form approach requires lots of memory and therefore can't solve larger problems. The main reason is that the algorithm doesn't detect the regular information set structure present in stacked matrix games, and also computes mixed strategies for all information sets, which may not be necessary. To overcome these problems [56] introduce a loss-less abstraction for games with certain regularity constraints and show that Nash equilibria found in the often much smaller game abstractions correspond to ones in the original game. General stacked matrix games don't fall into the game class considered in this paper, but the general idea of preprocessing games to transform them into smaller, equivalent ones may also apply to stacked matrix games.

5.1.3 Modelling RTS game combat

Battle unit management (also called *micromanagement*) is a core skill of successful human RTS game players and is vital to playing at a professional level. One of the top STARCRAFT players of all time, Jaedong, who is well known

for his excellent unit control, said in a recent interview: *"That micro made me different from everyone else in Brood War, and I won a lot of games on that micro alone"*.[1] It has also been proved to be decisive in the previous STARCRAFT AI competitions, with many battles between the top three AI agents being won or lost due to the quality of unit control. In this chapter we study small-scale battles we call *combats*, in which a small number of units interact in a small map region without obstructions.

In order to perform search for combat scenarios in STARCRAFT, we must construct a system which allows us to efficiently simulate the game itself. The BWAPI programming interface allows for interaction with the STARCRAFT interface, but unfortunately, it can only run the engine at 32 times "normal speed" and does not allow us to create and manipulate local state instances efficiently. As one search may simulate millions of moves, with each move having a duration of at least one simulation frame, it remains for us to construct an abstract model of STARCRAFT combat which is able to efficiently implement moves in a way that does not rely on simulating each in-game frame.

We will not dwelve into the details of the model we used to abstract STARCRAFT here, but more details can be found in Appendix A. Let us just recall that a game state correspond to a set of units for each player, that for each unit we know among other its position, hit points, delay before next attack or next move, as well as the damage per second it can perform. Finally, a player move is a set of unit actions, and each action can either be a move action, an attack action or a wait action.

5.2 Solution Concepts for Stacked-matrix Games

In this section, we recall a few solution concepts from game theory. We show how these solution concepts can be used to define perfect play and approximate play in stacked-matrix games.

5.2.1 Backward induction and Nash equilibria

A Nash equilibrium is a strategy profile for all players for which no player can increase his payoff by deviating unilaterally from his strategy. In the case of

[1] `http://www.teamliquid.net/forum/viewmessage.php?topic_id=339200`

zero-sum two-player games, all Nash equilibria result in the same payoff, called the *value* of the game. When faced with simultaneous actions, Nash equilibrium strategies are often mixed strategies in which actions are performed with certain probabilities (e.g., the only Nash equilibrium strategy for ROCK-PAPER-SCISSORS is playing Rock, Paper, and Scissors with probability $1/3$ each).

Two-player zero-sum games are often presented in normal-form which in a matrix lists payoffs for player *Max* for all action — or more generally pure strategy — pairs. Throughout this paper, player *Max* chooses rows, and player *Min* chooses columns. When working with normal-form games it is sometimes possible to simplify them based on action domination. This happens when no matter how the opponent acts, the payoff for some action a is always less or equal to the payoff for some other action b or a mixed strategy not containing a. In this situation there is no incentive to play action a and it can be ignored. The possibility of actions being dominated opens the door for pruning.

Example 12. Consider the game G presented in Example 11. It is possible to associate to each triple made of a state and a joint move a value based on the state that the joint move leads to. A graphical representation of G after it is solved is presented in Figure 5.2.

Note that, if the game is expressed implicitely, it is likely to be hard to solve. For intance, it was recently proved that deciding which player survives in combat games in which units can't even move is PSPACE-hard in general [52]. This means that playing stacked-matrix games optimally is computationally hard and that in practice we have to resort to approximations.

5.2.2 Game Theoretic Approximations

As mentioned above, combat games fall into the class of two-player zero-sum simultaneous move games. In this setting, the concepts of optimal play and game values are well defined, and the value $\mathrm{Nash}(G)$ of a game G (in view of the maximizing player *Max*) can be determined by using backward induction. However, as discussed earlier, this process can be very slow. Kovarsky and Buro (2005) describe how games with simultaneous moves can be sequentialized to make them amenable to fast alpha-beta tree search, trading optimality for speed. The idea is to replace simultaneous move states by two-level subtrees in

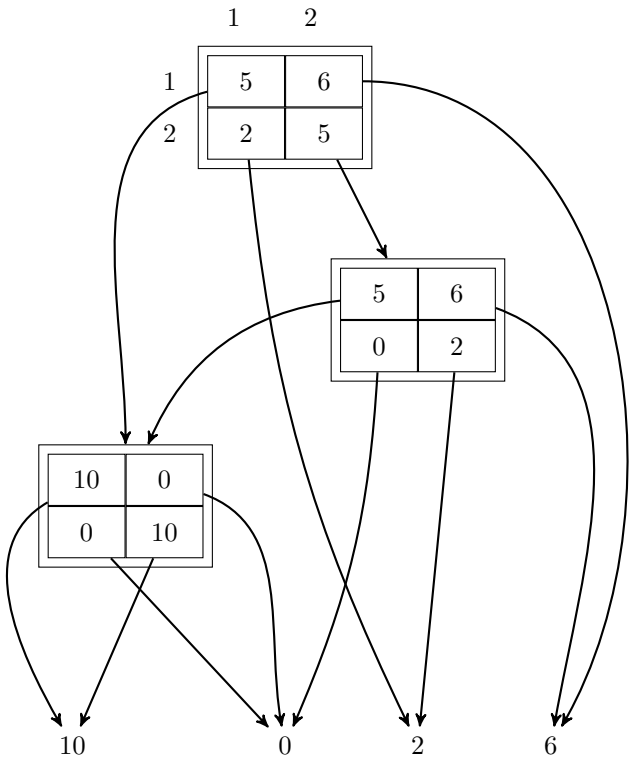


Figure 5.2: Graphical representation of game G from Example 11 once solved.

which players move in turn, maximizing respectively minimizing their utilities (Figure 5.3: Minmax and Maxmin). The value of the sequentialized games might be different from $\mathrm{Nash}(G)$ and it depends on the order we choose for the players in each state with simultaneous moves: if *Max* chooses his move first in each such state (Figure 5.3: Minmax), the value of the resulting game we call the *pure maxmin value* and denote it by $\mathrm{mini}(G)$. An elementary game theory result is that pure minmax and maxmin values are bounds for the true game value:

Proposition 35. *For stacked matrix games G, we have* $\mathrm{mini}(G) \leq \mathrm{Nash}(G) \leq$

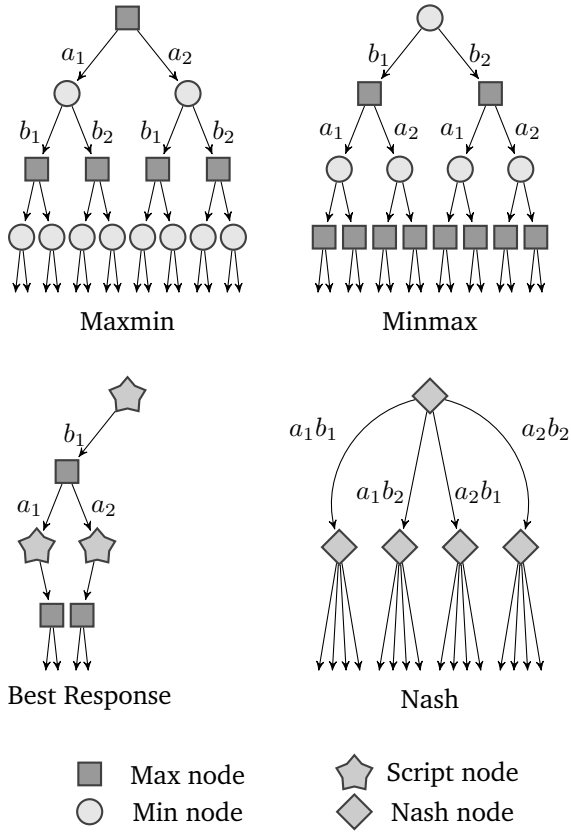
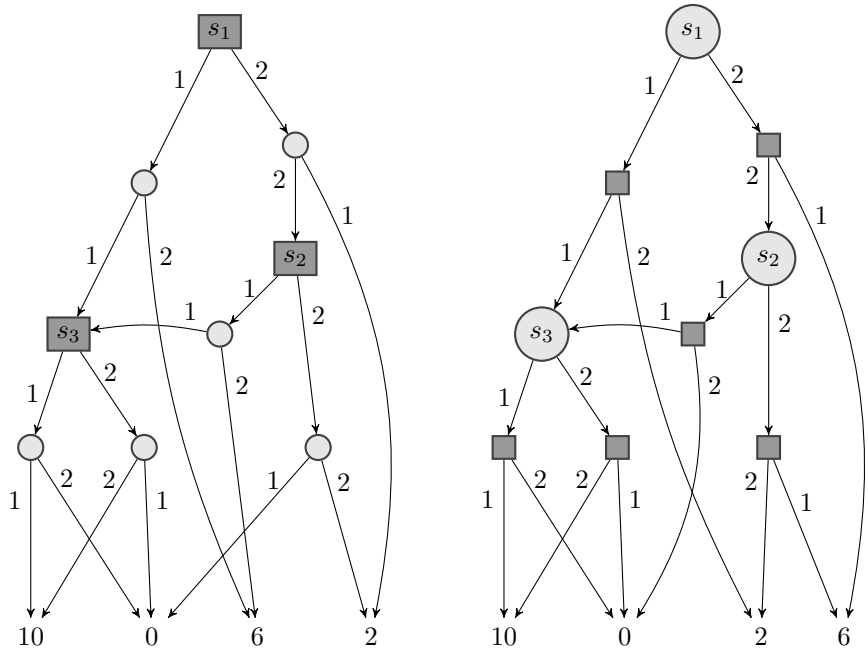


Figure 5.3: Graphical representations leading to the Maxmin, Minmax, Best Response, Nash solution concepts.

$\mathrm{maxi}(G)$*, and the inequalities are strict if and only if the game does not admit optimal pure strategies.*

Example 13. Consider the stacked-matrix game from Figure 5.1. The Maxmin and Minmax approximations are displayed in Figure 5.4.

These approximations are two-player multi-outcome games and can be solved within the framework described in Chapter 3. Take the game in Fig-



(a) Maxmin (b) Minmax

Figure 5.4: Graphical representations of the Maxmin and Minmax approximations for the game from Figure 5.1.

ure 5.4a, the state s_3 has value $\mathrm{mini}(s_3) = 0$, the state s_2 has value $\mathrm{mini}(s_2) = 0$, and so the state s_1 has value $\mathrm{mini}(s_1) = 0$ as well. In the minmax approximation Figure 5.4b, s_3 has value $\mathrm{maxi}(s_3) = 10$, s_2 has value $\mathrm{maxi}(s_2) = 6$, and s_1 has value $\mathrm{maxi}(s_1) = 6$ as well.

We know from Proposition 35 that these values can be used to bound the corresponding values in the original stacked matrix game. Thus $0 \leq \mathrm{Nash}(s_3) \leq 10$, $0 \leq \mathrm{Nash}(s_2) \leq 6$, and $0 \leq \mathrm{Nash}(s_1) \leq 6$. Looking back at Figure 5.2, we see that $\mathrm{Nash}(s_3) = 5$, $\mathrm{Nash}(s_2) = 5$, and $\mathrm{Nash}(s_1) = 5$ which confirms the above inequalities.

It is possible that there is no optimal pure strategy in a game with simulta-

neous moves, as ROCK-PAPER-SCISSORS proves. Less intuitively so, the need for randomized strategies also arises in combat games, even in cases with 2 vs. 2 immobile units [52]. To mitigate the potential unfairness caused by the Minmax and Maxmin game transformations, [79] propose the Random Alpha-Beta (RAB) algorithm. RAB is a Monte Carlo algorithm that repeatedly performs Alpha-Beta searches in transformed games where the player-to-move order is randomized in interior simultaneous move nodes. Once time runs out, the move with the highest total score at the root is chosen. [79] shows that RAB can outperform Alpha-Beta search on the Maxmin-transformed tree, using iterative deepening and a simple heuristic evaluation function. In our experiments, we will test the stripped down RAB version we call *RAB'*, which only runs Alpha-Beta once.

Another approach of mitigating unfairness is to alternate the player-to-move order in simultaneous move nodes on the way down the tree. We call this tree transformation *Alt*.

Because RAB' and the Alt transformation just change the player-to-move order, the following result on the value of the best RAB move ($\mathrm{rab}(G)$) and Alt move ($\mathrm{alter}(G)$) are easy to prove by induction on the tree height:

Proposition 36. *For stacked matrix game G, we have*

$$\mathrm{mini}(G) \leq \mathrm{rab}(G), \mathrm{alter}(G) \leq \mathrm{maxi}(G)$$

The proposed approximation methods are much faster than solving games by backward induction. However, the computed moves may be inferior. Section 5.5 we will see how they perform empirically.

5.3 Simultaneous Move Pruning

Table 5.2 summarizes related work of where pruning has been achieved in the context of MAE and clarifies where our contribution lies.

Example 14. Consider the game G presented in Example 11. After the game has been partially expanded and solved as shown in Figure 5.5, it becomes possible to compute the optimal strategies for both players at state s_1 without expanding state s_2. Indeed, any value for the joint move $(s_1, 2, 2)$ greater or equal to 2 results in move 1 being optimal for *Min*, and any value smaller or

Table 5.2: Pruning in Multi-Agent Environments

Sequential	Zero-sum	Agents	Pruning
Yes	Yes	Two	$\alpha\beta$
Yes	Yes	Any	[149]
Yes	No	-	[148]
No	Yes	Two	This chapter

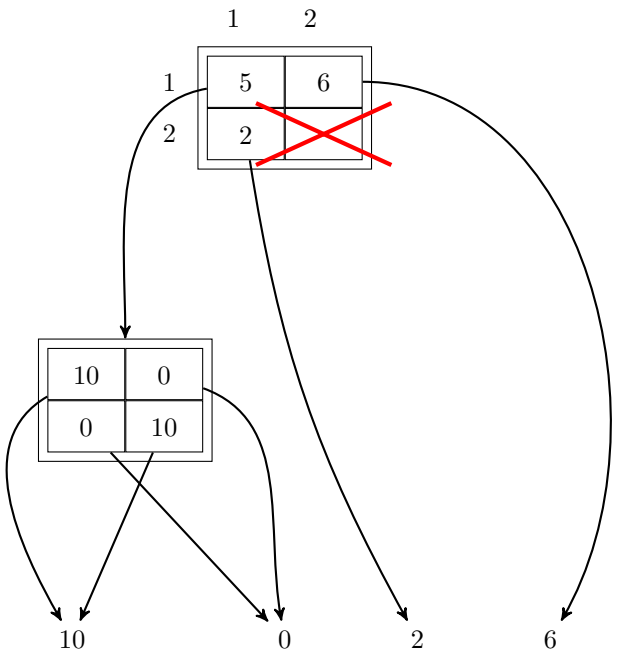


Figure 5.5: Graphical representation of game G from Example 11 partially expanded and solved and featuring an opportunity for pruning.

equal to 6 results in move 1 being optimal for *Max*. Thus, there is no value for the joint move $(s_1, 2, 2)$ that would make it belong to a Nash equilibrium.

The criterion we use for pruning is similar to that of the original Alpha-Beta algorithm: we prune sub-trees if we have a proof that they will under no

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{a-1} \\ x_{a+1} \\ \vdots \\ x_m \end{pmatrix}, P = \begin{pmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & & \vdots \\ p_{a-1,1} & \cdots & p_{a-1,n} \\ p_{a+1,1} & \cdots & p_{a+1,n} \\ \vdots & & \vdots \\ p_{m,1} & \cdots & p_{m,n} \end{pmatrix}, e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$f = \begin{pmatrix} o_{a,1} & \cdots & o_{a,n} \end{pmatrix}$$

$$x^t P \geq f, 0 \leq x \leq 1, \textstyle\sum_i x_i = 1$$

Figure 5.6: System of inequalities for deciding whether row action a is dominated. a is dominated and can be pruned if the system of inequalities is feasible.

circumstances improve upon the current guaranteed payoff assuming rational players.

Let s be a position in the game tree with m actions for *Max* and n actions for *Min*. For all $1 \leq i \leq m$ and $1 \leq j \leq n$, we call $s_{i,j}$ the position reached after joint action (i, j) is executed in s. We assume that the information we have gained so far about position $s_{i,j}$ is in the form of a pessimistic bound $p_{i,j}$ and an optimistic bound $o_{i,j}$ on the real value of $s_{i,j}$. For instance, if the value v of $s_{i,j}$ has been determined, we have $p_{i,j} = v = o_{i,j}$. If, however, no information about $s_{i,j}$ is known, we have $p_{i,j} =$ minval and $o_{i,j} =$ maxval.

To determine if a row action a can be safely pruned from the set of available *Max* actions in the presence of pessimistic payoff bounds $p_{i,j}$ and optimistic payoff bounds $o_{i,j}$ we use linear programming. A sufficient pruning condition is that action a is dominated by a mixed strategy excluding a. Using the given payoff bounds, we need to prove that there is a mixed strategy excluding action a that, when using pessimistic payoff bounds, dominates action a's optimistic payoff bounds. If such a mixed strategy exists then there is no need to consider action a, because a certain mixture of other actions is at least as good.

The system of inequalities (SI) in Figure 5.6 shows these calculations. If this system is feasible then action a can be pruned. Note that if $n = 1$, i.e., this state features a non-simultaneous action with *Max* to play, the SI reduces to the

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{a-1} \\ x_{a+1} \\ \vdots \\ x_m \end{pmatrix}, p = \begin{pmatrix} p_1 \\ \vdots \\ p_{a-1} \\ p_{a+1} \\ \vdots \\ p_m \end{pmatrix}, e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$x^t p \geq o_a, 0 \leq x \leq 1, \textstyle\sum_i x_i = 1$$

Figure 5.7: System of inequalities to decide if a row action a can be pruned when there is only one column action.

$$x = \begin{pmatrix} x_1 & \dots & x_{b-1} & x_{b+1} & \dots & x_m \end{pmatrix}$$

$$O = \begin{pmatrix} o_{1,1} & \dots & o_{1,b-1} & o_{1,b+1} & \dots & o_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ o_{m,1} & \dots & o_{m,b-1} & o_{m,b+1} & \dots & o_{m,n} \end{pmatrix}$$

$$f = \begin{pmatrix} p_{1,b} \\ \vdots \\ p_{m,b} \end{pmatrix}$$

$$e = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}$$

$$Ox^t \leq f, 0 \leq x \leq 1, \textstyle\sum_i x_i = 1$$

Figure 5.8: System of inequalities to decide if a column action b is dominated. b is dominated and can be pruned if the system of inequalities is feasible.

one shown in Figure 5.7. This SI is feasible if and only if there exists an action $a' \neq a$ such that $p_{a'} \geq o_a$. This is can be reformulated as pruning action a if $\max p_i \geq o_a$ which matches the pruning criterion in score bounded MCTS [26] exactly. The analogous SI for pruning dominated column actions is shown in Figure 5.8.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{a-1} \\ x_{a+1} \\ \vdots \\ x_m \\ x_{m+1} \end{pmatrix}, P = \begin{pmatrix} p_{1,1} & \dots & p_{1,b-1} & p_{1,b+1} & \dots & p_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{a-1,1} & \dots & p_{a-1,b-1} & p_{a-1,b+1} & \dots & p_{a-1,n} \\ p_{a+1,1} & \dots & p_{a+1,b-1} & p_{a+1,b+1} & \dots & p_{a+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{m,1} & \dots & p_{m,b-1} & p_{m,b+1} & \dots & p_{m,n} \\ \alpha & \dots & \alpha & \alpha & \dots & \alpha \end{pmatrix}, e = \begin{pmatrix} p_{1,b} \\ \vdots \\ p_{a-1,b} \\ p_{a+1,b} \\ \vdots \\ p_{m,b} \\ \alpha \end{pmatrix}$$

$$f = \begin{pmatrix} o_{a,1} & \dots & o_{a,b-1} & o_{a,b+1} & \dots & o_{a,n} \end{pmatrix}$$

$\alpha_{a,b} = \max x^t e$, subject to $x^t P \geq f$, $0 \leq x \leq 1$, $\sum_i x_i = 1$, or minval-1 if the LP is infeasible

Figure 5.9: Computing the pessimistic value $\alpha_{a,b}$

5.3.1 Alpha-Beta Search

Like the original Alpha-Beta algorithm, we traverse a given game tree in depth-first manner, for each position s using a lower bound α and an upper bound β on the value of s. As soon as we can prove that the value of s lies outside (α, β), we can prune the remaining positions below s and backtrack.

In this section we again assume that payoffs are given in view of row-player *Max* and that for each game state and player we have a bijection between legal moves and move indices starting at 1.

We begin by explaining how to determine the α and β bounds from pessimistic and optimistic value bounds. We then show how this computation can be integrated into a recursive depth-first search algorithm. Finally, we discuss some practical aspects.

5.3.2 Propagating Bounds

Let s be a position in the game tree and $A = \{1..m\}$ and $B = \{1..n\}$ the move sets for players *Max* and *Min*. For all $(i, j) \in A \times B$, we call $s_{i,j}$ the position reached after joint action (i, j) is executed in s. We assume that the information we have gained so far about position $s_{i,j}$ is in the form of a pessimistic bound $p_{i,j}$ and an optimistic bound $o_{i,j}$ on the real value of $s_{i,j}$. The default bound

$$x = \begin{pmatrix} x_1 & \dots & x_{b-1} & x_{b+1} & \dots & x_n & x_{n+1} \end{pmatrix}$$

$$O = \begin{pmatrix} o_{1,1} & \dots & o_{1,b-1} & o_{1,b+1} & \dots & o_{1,n} & \beta \\ \vdots & & \vdots & \vdots & & \vdots & \beta \\ o_{a-1,1} & \dots & o_{a-1,b-1} & o_{a-1,b+1} & \dots & o_{a-1,n} & \beta \\ o_{a+1,1} & \dots & o_{a+1,b-1} & o_{a+1,b+1} & \dots & o_{a+1,n} & \beta \\ \vdots & & \vdots & \vdots & & \vdots & \beta \\ o_{m,1} & \dots & o_{m,b-1} & o_{m,b+1} & \dots & o_{m,n} & \beta \end{pmatrix}, f = \begin{pmatrix} p_{1,b} \\ \vdots \\ p_{a-1,b} \\ p_{a+1,b} \\ \vdots \\ p_{m,b} \end{pmatrix}$$

$$e = \begin{pmatrix} o_{a,1} & \dots & o_{a,b-1} & o_{a,b+1} & \dots & o_{a,n} & \beta \end{pmatrix}$$

$\beta_{a,b} = \min ex^t$, subject to $Ox^t \leq f$, $0 \leq x^t \leq 1$, $\sum_i x_i = 1$, or maxval+1 if the LP is infeasible

Figure 5.10: Computing the optimistic value $\beta_{a,b}$

values are minval and maxval, respectively. Let $s_{a,b}$ be the next position to examine. We are interested in computing $\alpha_{s_{a,b}}$ and $\beta_{s_{a,b}}$ in terms of α, β (the value bounds for s), $p_{i,j}$ and $o_{i,j}$ for $(i,j) \in A \times B$. We first concentrate on computing $\alpha_{s_{a,b}}$, or $\alpha_{a,b}$ for short. $\beta_{a,b}$ can be derived analogously.

There are two reasons why we might not need to know the exact value of $s_{a,b}$, if it is rather small. Either we have proved that it is so small that a is dominated by a mixed strategy not containing a (shallow pruning), or it is so small that as a result we can prove that the value of s is smaller than α (deep pruning). We can combine both arguments into one LP by adding an artificial action $m+1$ for *Max* that corresponds to *Max* deviating earlier. This action guarantees a score of at least α, i.e., $p_{m+1,j} = \alpha$ for all $j \in B$. We can now restrict ourselves to determining under which condition action a would be dominated by a mixed strategy of actions $M := \{1, \dots, m+1\} \backslash \{a\}$. To guarantee soundness, we need to look at the situation where a is least expected to be pruned, i.e., when the values of positions $s_{a,j}$ reach their optimistic bounds $o_{a,j}$ and for every other action $i \neq a$, the values of positions $s_{i,j}$ reach their pessimistic bounds $p_{i,j}$.

Consider the set of mixed strategies D dominating a on every column but b,

i.e.,

$$D = \{x \in \mathbb{R}^m_{\geq 0} \mid \sum_i x_i = 1, \forall j \neq b : \sum_{i \in M} x_i p_{i,j} \geq o_{a,j}\} \tag{5.2}$$

Action a is dominated if and only if a is dominated on column b by a strategy in D. I.e., action a is dominated if and only if value v of $s_{a,b}$ satisfies:

$$\exists x \in D : \sum_{i \in M} x_i p_{i,b} \geq v \tag{5.3}$$

If D is non-empty, to have the tightest $\alpha_{a,b}$ possible, we maximize over such values:

$$\alpha_{a,b} = \max_{x \in D} \sum_{i \in M} x_i p_{i,b} \tag{5.4}$$

Otherwise, if D is empty, $s_{a,b}$ can't be bound from below and we set $\alpha_{a,b} =$ minval.

This process can be directly translated into the LP presented in Figure 5.9. Similarly, the bound $\beta_{s_{a,b}}$ is defined as the objective value of the LP shown in Figure 5.10.

5.3.3 Main Algorithm

Algorithm 10 describes how our simultaneous move pruning can be incorporated in a depth-first search algorithm by looping through all joint action pairs first checking trivial exit conditions and if these fail, proceeding with computing optimistic and pessimistic bounds for the entry in questions, and then recursively computing the entry value. We call this procedure Simultaneous Move Alpha-Beta (SMAB) Search.

Theorem 8. *When SMAB is called with s, α, β and $\alpha < \beta$ ...*

1. *... it runs in weakly polynomial time in the size of the game tree rooted in s.*
2. *... and returns $v \leq \alpha$, then value(s) $\leq v$.*
3. *... and returns $v \geq \beta$, then value(s) $\geq v$.*
4. *... and returns $\alpha < v < \beta$, then value(s) $= v$.*

Proof sketch.

Algorithm 10: Pseudo-code for simultaneous move Alpha-Beta search. Function $\mathrm{Nash}(X)$ computes the Nash equilibrium value of normal-form game payoff matrix X for row player *Max*.

```
SMAB(state s, lower bound α, upper bound β)
  if s ∈ F then return σ(s)
  else
    p_{i,j} ← b for i ∈ L_1(s), j ∈ L_2(s)
    o_{i,j} ← a for i ∈ L_1(s), j ∈ L_2(s)
    Let P denote the matrix formed by all p_{i,j}
    Let O denote the matrix formed by all o_{i,j}
    foreach (a, b) ∈ L_1(s) × L_2(s) do
      if row a and column b are not dominated then
        Let α_{a,b} as defined in Fig. 5.9 restricted to non-dominated
        actions
        Let β_{a,b} as defined in Fig. 5.10 restricted to non-dominated
        actions
        s_{a,b} ← the state reached after applying (a, b) to s
        if α_{a,b} ≥ β_{a,b} then
          v_{a,b} ← SMAB(s_{a,b}, α_{a,b}, α_{a,b} + ε)
          if v_{a,b} ≤ α_{a,b} then a is dominated
          else b is dominated
        else
          v_{a,b} ← SMAB(s_{a,b}, α_{a,b}, β_{a,b})
          if v_{a,b} ≤ α_{a,b} then a is dominated
          else if v_{a,b} ≥ β_{a,b} then b is dominated
          else p_{a,b} ← v_{a,b}; o_{a,b} ← v_{a,b}
    return Nash(P restricted to non-dominated actions)
```

1.: Weakly polynomial run-time in the sub-tree size can be shown by induction on the tree height using the fact that LPs can be solved by interior point methods in weakly polynomial time.

2.,3.,4.: Induction on tree height h. For $h = 0$, SMAB immediately returns the true value. Thus, properties 2.-4. hold. Now we assume they hold for all heights $h \leq k$ and s has height $k + 1$ and proceed with an induction on the number of inner loop iterations claiming that P and O are correctly

updated in each step (using the derivations in the previous subsection and the main induction hypothesis) and if line 28 is reached, properties 2.-4. hold.

□

5.3.4 Ordering Move Pairs

Heuristics can be used to initialize $(p_{i,j}, o_{i,j})$, given that they have the admissibility property with regards to the bound they are applied to. As an example, we might in some game know from the material strength on the board in some state that we are guaranteed at least a draw, allowing us to initialize the pessimistic value to a draw. Similarly, we should be able to set the optimistic value to a draw if the opponent is equally up in material.

Additionally, the order in which the pairs (a, b) will be visited in Algorithm 10 can dramatically affect the amount of pruning. This problem can be decomposed into two parts. *Move ordering* in which the individual moves are ordered and *cell ordering* in which the joint moves are ordered based on the order of the individual moves. Formally, move ordering means endowing the sets A and B with total orders $<_A$ and $<_B$ and cell ordering is the construction of a total order for $A \times B$ based on $<_A$ and $<_B$. For instance, the lexicographical ordering is a possible cell ordering: (a_1, b_1) will be explored before (a_2, b_2) iff $a_1 <_A a_2$ or $a_1 = a_2$ and $b_1 < b_2$. We will discuss heuristic cell orderings in the next section.

5.4 Fast approximate search for combat games

In the previous section we discussed multiple game transformations that would allow us to find solutions by using backward induction. However, when playing RTS games the real-time constraints are harsh. Often, decisions must be made during a single simulation frame, which can be 50 ms or shorter. Therefore, computing optimal moves is impossible for all but the smallest settings and we need to settle for approximate solutions: we trade optimality for speed and hope that the algorithms we propose defeat the state of the art AI systems for combat games.

The common approach is to declare nodes to be leaf nodes once a certain depth limit is reached. In leaf nodes *Max*'s utility is then estimated by calling an *evaluation function*, and this value is propagated up the tree like true terminal node utilities.

In the following subsections we will first adapt the Alpha-Beta search algorithm to combat games by handling durative moves explicitly and then present a series of previously known and new evaluation functions.

5.4.1 Scripted behaviors

The simplest approach, and the one most commonly used in video game AI systems, is to define static behaviors via AI scripts. Their main advantage is computation speed, but they often lack foresight, which makes them vulnerable against search-based methods, as we will see in Section 5.5, where we will evaluate the following simple combat AI scripts:

- The *Random* strategy picks legal moves with uniform probability.
- Using the *Attack-Closest* strategy units will attack the closest opponent unit within weapon's range if it can currently fire. Otherwise, if it is within range of an enemy but is reloading, it will wait in-place until it has reloaded. If it is not in range of any enemy, it will move toward the closest enemy a fixed distance.
- The *Attack-Weakest* strategy is similar to Attack-Closest, except units attack an opponent unit with the lowest hp within range when able.
- Using the *Kiting* strategy is similar to Attack-Closest, except it will move a fixed distance away from the closest enemy when it is unable to fire.

The Attack-Closest script was used in second-place entry *UAlbertaBot* in the 2010 AIIDE STARCRAFT AI competition, whereas *Skynet*, the winning entry, used a behavior similar to Kiting. The scripts presented so far are quite basic but we can add a few smarter ones to our set of scripts to test.

- The *Attack-Value* strategy is similar to Attack-Closest, except units attack an opponent unit u with the highest $\mathrm{dpf}(u)/\mathrm{hp}(u)$ value within range when able. This choice leads to optimal play in 1 vs. n scenarios [52].

- The *No Overkill/Attack Value (NOK-AV)* strategy is similar to Attack-Value, except units will not attack an enemy unit which has been assigned lethal damage this round. It will instead choose the next priority target, or wait if one does not exist.

- Using the *Kiting-AV* strategy is similar to Kiting, except it will choose an attack target similar to Attack-Value.

Most scripts we described make decisions on an individual unit basis, with some creating the illusion of unit collaboration (by concentrating fire on closest, weakest, or most-valuable units). NOK-AV is the only script in our set that exhibits true collaborative behaviour by sharing information about unit targeting.

5.4.2 Alpha-Beta Search with Durative Moves

Consider Figure 5.11 which displays a typical path in the sequentialized game tree. Because of the weapon $\mathrm{cooldown}$ and the space granularity, battle games exhibit numerous *durative moves*. Indeed, there are many time steps where the only move for a player is just pass, since all the units are currently unable to perform an action. Thus, non-trivial decision points for players do not occur on every frame.

Given a player p in a state s, define the next time where p is next able to do a non-pass move by $\tau(s,p) = \min_{u \in s.U_p}(u.\mathrm{t}_a, u.\mathrm{t}_m)$. Note that for any time step t such that $s.\mathrm{t} < t < \min(\tau(s,\mathrm{Max}), \tau(s,\mathrm{Min}))$, players cannot perform any move but pass. It is therefore possible to shortcut many trivial decision points between $s.\mathrm{t}$ and $\min(\tau(s,\mathrm{Max}), \tau(s,\mathrm{Min}))$.

Assume an evaluation function has been picked, and remaining simultaneous choices are sequentialized as suggested above. It is then possible to adapt the Alpha-Beta algorithm to take advantage of durative moves as presented in Algorithm 11

We use the *terminal(s, d)* function to decide when to call the evaluation function. It is parametrized by a maximal depth $d_{\max}$ and a maximal time $t_{\max}$ and return "true" if s is a terminal position or $d \geq d_{\max}$ or $s.\mathrm{t} \geq t_{\max}$.

The third argument to the ABCD algorithm is used to handle the *delayed action effect* mechanism for sequentialized simultaneous moves. If the state does not correspond to a simultaneous decision, m_0 holds a dummy value $\emptyset$.

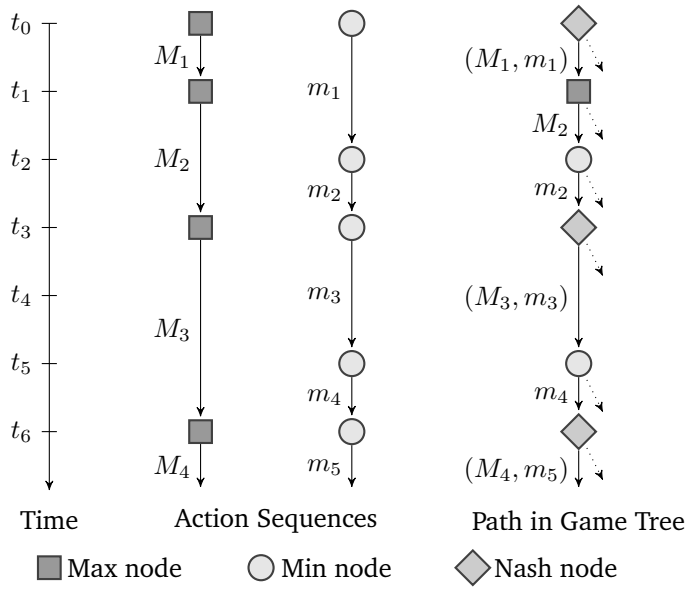


Figure 5.11: Durations

Otherwise, we apply the effects of m_0 after move m is generated because m_0 should not affect the generation of the complementary moves.

5.4.3 Evaluation Functions

A straight-forward evaluation function for combat games is the hitpoint-total differential (Equation 5.5) which, however, does not take into account other unit properties, such as damage values and cooldown periods.

$$e(s) = \sum_{u \in U_1} \mathrm{hp}(u) - \sum_{u \in U_2} \mathrm{hp}(u) \tag{5.5}$$

Kovarsky and Buro (2005) propose an evaluation based on the life-time damage a unit can inflict, which is proportional to its hp times its damage-per-frame ratio:

Algorithm 11: Alpha-Beta (Considering Durations) (ABCD)

```
abcd(s, d, m_0, α, β)
  if computationTime.elapsed then return timeout
  else if s ∈ F or d = 0 then return ζ(s)
  τ ← s.playerToMove(policy)
  while m ← s.nextMove(τ) do
    if s.bothCanMove and m_0 = ∅ and d ≠ 1 then
      v ←abcd(s, d − 1, m, α, β)
    else
      s' ← s
      if m_0 ≠ ∅ then s'.doMove(m_0)
      s'.doMove(m)
      v ←abcd(s', d − 1, ∅, α, β)
    if τ = Max and v > α then α ← v
    if τ = Min and v < β then β ← v
    if α ≥ β then break
  if τ = Max then return α
  else return β
```

$$\mathrm{dpf}(u) = \frac{\mathrm{damage}(w(u))}{\mathrm{cooldown}(w(u))} \tag{5.6}$$

$$\mathrm{LTD}(s) = \sum_{u \in U_1} \mathrm{hp}(u) \cdot \mathrm{dpf}(u) - \sum_{u \in U_2} \mathrm{hp}(u) \cdot \mathrm{dpf}(u) \tag{5.7}$$

A second related evaluation function propose favours uniform hp distributions [79]:

$$\mathrm{LTD2}(s) = \sum_{u \in U_1} \sqrt{\mathrm{hp}(u)} \cdot \mathrm{dpf}(u) - \sum_{u \in U_2} \sqrt{\mathrm{hp}(u)} \cdot \mathrm{dpf}(u) \tag{5.8}$$

While these evaluation functions are exact for terminal positions, they can be drastically inaccurate for many non-terminal positions. To improve state evaluation by also taking other unit properties such as speed and weapon range into account, we can try to simulate a game and use the outcome as an estimate of the utility of its starting position. This idea is known as *performing a playout*

in game tree search and is actually a fundamental part of MCTS algorithms which have revolutionized computer GO [40]. However, there are differences between the playouts we advocate for combat games and previous work on GO and HEX: the playout policies we use here are deterministic and we are not using MCTS or a best-first search algorithm, but rather depth-first search.

5.4.4 Move Ordering

It is well-known in the game AI research community that a good move ordering fosters the performance of the Alpha-Beta algorithm. When Transposition Tables (TTs) and iterative deepening are used, reusing previous search results can improve the move ordering. Suppose a position p needs to be searched at depth d and was already searched at depth d'. If $d \leq d'$, the value of the previous search is sufficiently accurate and there is no need for an additional search on p. Otherwise, a deeper search is needed, but we can explore the previously found best move first and hope for more pruning.

When no TT information is available, we can use scripted strategies to suggest moves. We call this new heuristic *scripted move ordering*. Note that this heuristic could also be used in standard sequential games like CHESS. We believe the reason it has not been investigated closely in those contexts is the lack of high quality scripted strategies.

5.5 Experiments

5.5.1 Solving GOOFSPIEL

As a test case we implemented SMAB pruning for the game of GOOFSPIEL. The following experimental results were obtained running OCaml 3.11.2, g++ 4.5.2, and the glpk 4.43 LP-solver under Ubuntu on a laptop with Intel T3400 CPU at 2.2 GHz.

The game GOOFSPIEL [117, 142] uses cards in three suits. In the version we use, each player has all the cards of a single suit and the remaining suit is stacked on the table face up in a pre-defined order. On each turn both players simultaneously play a card from their hand and the higher card wins its player the top card from the table. If the played cards are of equal value the table card is discarded. When all cards have been played the winner is the player whose

1	2	3	4	5
6	10	11	12	13
7	14	17	18	19
8	15	20	22	23
9	16	21	24	25

Figure 5.12: L-shaped cell ordering for 5×5 matrices.

accumulated table cards sum up to a higher value. We used games with various number of cards per suit to monitor how the pruning efficiency develops with increasing game-tree sizes.

We use a cell ordering that strives to keep a balance between the number of rows filled and the number of columns filled. We call it *L-shaped* and it can be seen as the lexicographical ordering over tuples $(\min\{a, b\}, a, b)$. Its application to 5×5 matrix is described in Figure 5.12. In all of our preliminary experiments, the L-shaped ordering proved to lead to earlier and more pruning than the natural lexicographical orderings.

To save some calculations, it is possible to skip the LP computations for some cells and directly set the corresponding α and β bounds to $(b - 1)$ and $(a + 1)$, respectively. On the one hand, if the computed bounds wouldn't have enabled much pruning, then using the default bounds instead allows to save some time. On the other hand, if too many bounds are loose, there will be superfluous computations in prunable subtrees.

To express this tradeoff, we introduce the *early bound skipping* heuristic. This heuristic is parameterized by an integer s and consists in skipping the LP-based computations of the α and β bounds as long as the matrix does not have at least s rows and s columns completely filled. For instance, if we use this heuristic together with the L-shaped ordering on a 5×5 matrix with parameter $s = 1$, no LP computation takes place for the bounds of the first 9 cells.

In our backward induction implementation that recursively solves subgames in depth-first fashion, we used one LP call per non-terminal node expansion. Table 5.3 shows the number of non-terminal node expansions/LP calls, the total

Table 5.3: Solving GOOFSPIEL with backward induction.

size	nodes (= LP calls)	total time	LP time
4	109	0.008	0.004
5	1926	0.188	0.136
6	58173	5.588	4.200
7	2578710	247.159	184.616

Table 5.4: Solving GOOFSPIEL with a sequence form solver.

size	memory	time
4	8 MB	< 1 s
5	43 MB	152 s
6	$>$ 2 GB	> 177 s

Table 5.5: Solving GOOFSPIEL with SMAB.

size	nodes	LP calls	total time	LP time	s
4	55	265	0.020	0.016	0
4	59	171	0.012	0.012	1
4	70	147	0.012	0.012	2
5	516	2794	0.216	0.148	0
5	630	**1897**	**0.168**	0.128	1
5	1003	1919	0.184	0.152	2
6	13560	74700	5.900	4.568	0
6	18212	**55462**	**4.980**	3.852	1
6	30575	57335	5.536	4.192	2
7	757699	4074729	324.352	245.295	0
7	949521	2857133	259.716	197.700	1
7	1380564	2498366	241.735	182.463	2
7	1734798	**2452624**	**237.903**	177.411	3
7	1881065	2583307	253.476	188.276	4

time spent running the algorithm, and the time spent specifically solving LPs.

Table 5.5 shows the same information for SMAB using L-shaped ordering and early bound skipping parameterized by s. This table has separate columns for the number of non-terminal node expansions and the number of calls to the LP solver as they are not equal in the case of SMAB.

Table 5.4 shows the memory and time needed to solve GOOFSPIEL using a sequence form solver based on based on [77]. The algorithm needs a huge amount of memory to solve even a moderate size instance of GOOFSPIEL. The backward induction and the SMAB implementations, on the contrary, never needed more than 60 MB of memory. This difference is expected as the backward induction and SMAB are depth-first search algorithms solving hundreds of thousands of relatively small LPs while the sequence form algorithm solves a single large LP.

5.5.2 Real-time Strategy games

We implemented the proposed combat model, the scripted strategies, the new ABCD algorithm, and various tree transformations. We then ran experiments to measure 1) the influence of the suggested search enhancements for determining the best search configuration, and 2) the real-time exploitability of scripted strategies.

Because of time constraints, we were only able to test the following tree transformations: Alt, Alt', and RAB', where Alt' in simultaneous move nodes selects the player that acted last, and RAB' selects the player to move like RAB, but only completes one Alpha-Beta search.

Setup The combat scenarios we used for the experiments involved equally sized armies of n versus n units, where n varied from 2 to 8. 1 versus 1 scenarios were omitted due to over 95% of them resulting in draws. Four different army types were constructed to mimic various combat scenarios. These armies were: *Marine* Only, *Marine* + *Zergling*, *Dragoon* + *Zealot*, and *Dragoon* + *Marine*. Armies consisted of all possible combinations of the listed unit type with up to 4 of each, for a maximum army size of 8 units. Each unit in the army was given to player *Max* at random starting position (x, y) within 256 pixels of the origin, and to player *Min*at position $(-x, -y)$, which guaranteed symmetric start locations about the origin. Once combat began, units were allowed to

move infinitely within the plane. Unit movement was limited to up, down, left, right at 15 pixel increments, which is equal to the smallest attack range of any unit in our tests.

These settings ensured that the Nash value of the starting position was always 0.5. If the battle did not end in one player being eliminated after 500 actions, the simulation was halted and the final state evaluated with LTD. For instance, in a match between a player p_1 and an opponent p_2, we would count the number of wins by p_1, w, and number of draws, d, over n games and compute $r = (w + d/2)/n$. Both players perform equally well in this match if $r \approx 0.5$.

As the 2011 STARCRAFT AI Competition allowed for 50 ms of processing per game logic frame, we gave each search episode a time limit of 5 ms. This simulates the real-time nature of RTS combat, while leaving plenty of time (45 ms) for other processing which may have been needed for other computations.

Experiments were run single-threaded on an Intel Core i7 2.67 GHz CPU with 24 GB of 1600 MHz DDR3 RAM using the Windows 7 64 bit operating system and Visual C++ 2010. A transposition table of 5 million entries (20 bytes each) was used. Due to the depth-first search nature of the algorithm, very little additional memory is required to facilitate search. Each result table entry is the result of playing 365 games, each with random symmetric starting positions.

Influence of the Search Settings To measure the impact of certain search parameters, we perform experiments using two methods of comparison. The first method plays static scripted opponents vs. ABCD with various settings, which are then compared. The second method plays ABCD vs. ABCD with different settings for each player.

We start by studying the influence of the evaluation function selection on the search performance (see Section 5.4.3). Preliminary experiments revealed that using NOK-AV for the playouts was significantly better than using any of the other scripted strategies. The playout-based evaluation function will therefore always use the NOK-AV script.

We now present the performance of various settings for the search against script-based opponents (Table 5.6) and search-based opponents (Table 5.7). In Table 5.6, the Alt sequentialization is used among the first three settings

Table 5.6: ABCD vs. Script - scores for various settings

Opponent	ABCD Search Setting				
	Alt LTD	Alt LTD2	Alt	Alt'	RAB'
			NOK-AV Playout		
Random	0.99	0.98	1.00	1.00	1.00
Kite	0.70	0.79	0.93	0.93	0.92
Kite-AV	0.69	0.81	0.92	0.96	0.92
Closest	0.59	0.85	0.92	0.92	0.93
Weakest	0.41	0.76	0.91	0.91	0.89
AV	0.42	0.76	0.90	0.90	0.91
NOK-AV	0.32	0.64	0.87	0.87	0.82
Average	0.59	0.80	0.92	0.92	0.91

Table 5.7: Playout-based ABCD performance

Opponent	Alt	Alt'	RAB'
	NOK-AV Playout		
Alt-NOK-AV		0.47	0.46
Alt'-NOK-AV	0.53		0.46
RAB'-NOK-AV	0.54	0.54	
Average	0.54	0.51	0.46

which allow to compare the leaf evaluations functions LTD, LTD2, and playout-based. The leaf evaluation based on NOK-AV playouts is used for the last three settings which allow to compare the sequentialization alternatives described in Subsection 5.2.2.

We can see based on the first three settings that doing a search based on a good playout policy leads to much better performance than doing a search based on a static evaluation function. The search based on the NOK-AV playout strategy is indeed dominating the searches based on LTD and LTD2 against any opponent tested. We can also see based on the last three settings that the Alt and Alt' sequentializations lead to better results than RAB'.

Table 5.8: Real-time exploitability of scripted strategies.

Random	Weakest	Closest	AV	Kiter	Kite-AV	NOK-AV
1.00	0.98	0.98	0.98	0.97	0.97	0.95

Estimating the Quality of Scripts The quality of scripted strategies can be measured in at least two ways: the simplest approach is to run the script against multiple opponents and average the results. To this end, we can use the data presented in Table 5.6 to conclude that NOK-AV is the best script in our set. Alternatively, we can measure the exploitability of scripted strategies by determining the score a theoretically optimal best-response-strategy would achieve against the script. However, such strategies are hard to compute in general. Looking forward to modelling and exploiting opponents, we would like to approximate best-response strategies quickly, possibly within one game simulation frame. This can be accomplished by replacing one player in ABCD by the script in question and then run ABCD to find approximate best-response moves. The obtained tournament result we call the *real-time exploitability* of the given script. It constitutes a lower bound (in expectation) on the true exploitability and tells us about the risk of being exploited by an adaptive player. Table 5.8 lists the real-time exploitability of various scripted strategies. Again, the NOK-AV strategy prevails, but the high value suggests that there is room for improvement.

5.6 Conclusion and Future Work

We have shown that it is possible to extend Alpha-Beta pruning to include simultaneous move games and that our SMAB pruning procedure can reduce the node count and run-time when solving non-trivial games. In the reported experiments we used a fixed move ordering and a fixed cell ordering. The results show a considerable drop in node expansions, even though not nearly as much as with Alpha-Beta in the sequential setting, but certainly enough to be very promising. Still, this threshold is not high and with increasing game size the run-time appears to be increasingly improving. The pruning criterion we propose is sound, but it only allows us to prune provably dominated actions.

Example 15 shows that it sometimes happen that some strategies are not part of any Nash equilibria, but cannot be eliminated by iterative dominance. As a result, some actions which are irrelevant may not get pruned by our method. SMAB yields considerable savings in practice, but this example shows that there is room for even more pruning.

Example 15. The following game has a unique Nash equilibrium at (A_2, B_2), but no action is dominated.

	B_1	B_2	B_3
A_1	6	1	0
A_2	3	3	3
A_3	0	1	6

It will be interesting to see how SMAB pruning performs in other domains and it can also be applied to MCTS which has become the state-of-the-art algorithmic framework for computer GO and the GENERAL GAME PLAYING competition. A natural candidate is to extend the score bounded MCTS framework that we described earlier.

In our implementation we just used a naive move ordering scheme. However, simultaneous moves offer some interesting opportunities for improvements. As each individual action is considered more than once in a state, we get some information on them before their pairings are fully enumerated. The question is whether we can use this information to order the actions such that the efficiency of the pruning increases, like it does for sequential Alpha-Beta search.

Finally, it may be possible to establish the minimal number of node expansions when solving certain classes of stacked matrix games with depth-first search algorithms in general, or SMAB in particular.

In this chapter we have also presented a framework for fast Alpha-Beta search for RTS game combat scenarios of up to 8 vs. 8 units and evaluated it under harsh real-time conditions. Our method is based on an efficient combat game abstraction model that captures important RTS game features, including unit motion, an Alpha-Beta search variant (ABCD) that can deal with durative moves and various tree transformations, and a novel way of using scripted strategies for move ordering and depth-first-search state evaluation via playouts.

The experimental results are encouraging. Our search, when using only 5 ms per episode, defeats standard AI scripts as well as more advanced scripts that exhibit kiting behaviour and minimize overkill. The prospect of opponent modelling for exploiting scripted opponents is even greater: the practical exploitability results indicate large win margins best-response ABCD can achieve if the opponent executes any of the tested combat scripts.

The ultimate goal of this line of research is to handle larger combat scenarios with more than 20 units on each side in real-time. The enormous state and move complexity, however, prevents us from applying heuristic search directly, and we therefore will have to find spatial and unit group abstractions that reduce the size of the state space so that heuristic search can produce meaningful results in real-time. Balla and Fern (2009) present initial research in this direction, but their Upper Confidence bound for Trees (UCT)-based solution is rather slow and depends on pre-assigned unit groups.

Our next steps will be to integrate ABCD search into a STARCRAFT AI competition entry to gauge its performance against previous year's participants, to refine our combat model if needed, to add opponent modelling and best-response-ABCD to counter inferred opponent combat policies, and then to tackle more complex combat scenarios.

6 Conclusion

In this thesis, we have proposed a generic BFS framework for two-outcome games and we have adapted and extended it to multi-outcome games and MMLK model checking. This gives access to generalizations of PNS and MCTS Solver for multi-outcome games and for MMLK model checking. An attractive direction for future work would be to express a similar BFS framework for other classes of multi-agent systems. Candidate classes comprise simple stochastic games or competitive Markov Decision Processes (MDPs) [35, 36, 44], and stacked-matrix games as defined in Chapter 5.

We have generalized alpha-beta pruning to stacked-matrix games. Developing a safe pruning criterion for the general class of Multi-Agent Environment as defined by Schiffel and Thielscher [138] seems accessible now, as it only requires combining our approach with that of Sturtevant for non-zero and multi-player games [147, 148].

The approach that we have taken in Chapter 2, 3, and 5 make an implicit closed-world assumption: we assumed that computing the game-theoretic value of a position was all the information we could ever need. Combinatorial Game Theory goes a step further as it considers it possible that the game could be just a small part of a bigger game [1]. In such a setting, it is possibe that a player plays twice in a row in a subgame if the opponent plays in another subgame. As a result, there is more information to extract from a position than its minimax value. The maximum amount of information a position contains is called the *canonical form* and algorithms have been developed to compute the canonical form of any two-player combinatorial game.

However, these algorithms are quite slow and sometimes spend time com-

puting redundant or useless information. Providing a safe pruning mechanism à la alpha-beta would certainly enable much faster computations in certain cases. We can also envision adapting the BFS framework presented in this thesis to the computation of canonical forms in combinatorial game theory.

A Combat game abstract model

To fully simulate RTS game combat, our model is comprised of three main components: states, units, and moves.

State $s = \langle \mathrm{t}, U_1, U_2 \rangle$

- Current game time t
- Sets of units U_i under control of player i

Unit $u = \langle p, \mathrm{hp}, \mathrm{t}_a, \mathrm{t}_m, v, w \rangle$

- Position $p = (x, y)$ in $\mathbb{R}^2$
- Current hit points hp
- Time step when unit can next attack t_a, or move t_m
- Maximum unit velocity v
- Weapon properties $w = \langle \mathrm{damage}, \mathrm{cooldown} \rangle$

Move $m = \{a_0, \ldots, a_k\}$ which is a combination of unit actions $a_i = \langle u, \mathrm{type}, \mathrm{target}, t \rangle$, with

- Unit u to perform this action
- The type type of action to be performed:

  Attack unit target
  Move u to position target
  Wait until time t

Given a state s containing unit u, we generate legal unit actions as follows: if $u.\mathrm{t}_a \leq s.\mathrm{t}$ then u may *attack* any target in its range, if $u.\mathrm{t}_m \leq s.\mathrm{t}$ then u may *move* in any legal direction, if $u.\mathrm{t}_m \leq s.\mathrm{t} < u.\mathrm{t}_a$ then u may *wait* until $u.\mathrm{t}_a$. If both $u.\mathrm{t}_a$ and $u.\mathrm{t}_m$ are $> s.\mathrm{t}$ then a unit is said to have no legal actions. A legal player move is then a set of all combinations of one legal unit action from each unit a player controls.

Unlike strict alternating move games like chess, our model's moves have durations based on individual unit properties, so either player (or both) may be able to move at a given state. We define the player to move next as the one which contains the unit with the minimum time for which it can attack or move.

While the mathematical model we propose does not exactly match the combat mechanics of STARCRAFT it captures essential features. Because we don't have access to STARCRAFT's source code, we can only try to infer missing features based on game play observations:

- no spell casting (e.g., immobilization, area effects)
- no hit point or shield regeneration
- no travel time for projectiles
- no unit collisions
- no unit acceleration, deceleration or turning
- no fog of war

Quite a few STARCRAFT AI competition entries are designed with a strong focus on early game play (rushing). For those programs some of the listed limitations, such as single weapons and spell casting, are immaterial because they become important only in later game phases. The utility of adding others, such as dealing with unit collisions and acceleration, will have to be determined once our search technique becomes adopted.

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